International Journal of Mathematical Archive-3(8), 2012, 3197-3211
IJMA Available online through www.ijma.info ISSN 2229 – 5046

ON THE STABILITY OF A FOUR SPECIES SYN ECO-SYSTEM WITH COMMENSAL PREY-PREDATOR PAIR WITH PREY-PREDATOR PAIR OF HOSTS-II

(1st Level Prey-Predator Washed Out States)

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(Received on: 07-07-12; Accepted on: 30-07-12)

ABSTRACT

The present paper is devoted to an investigation on a Four Species (S_1, S_2, S_3, S_4) Syn Eco-System with Commensal Prey-Predator pair with Prey-Predator pair of Hosts [Prey (S_1) and Predator (S_2) Washed Out States]. The System comprises of a Prey (S_1), a Predator (S_2) that survives upon S_1, two Hosts S_3 and S_4 for which S_1, S_2 are Commensal respectively i.e., S_3 and S_4 benefit S_1 and S_2 respectively, without getting effected either positively or adversely. Further S_3 is Prey for S_4 and S_4 is Predator for S_3. The pair (S_1, S_2) may be referred as 1st level Prey-Predator and the pair (S_3, S_4) the 2nd level Prey-Predator. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of three of these sixteen equilibrium points: 1st Level Prey-Predator Washed Out States are established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories are illustrated.

Keywords: *Commensal, Eco-System, Equillibrium point, Host, Prey, Predator, Stable, Trajectories.*

AMS Classification: *92D25, 92D40.*

1. INRTRODUCTION

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [1] and by Volterra [2]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Meyer [3], Kushing [4], Paul colinvaux [5], Kapur [6] etc.The ecological interactions can be broadly classifiedas Prey-predation, Competition, Commensalim, Ammensalism, Neutralism and so on. N.C. Srinivas [7] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later, Lakshminarayan [8], Lakshminarayan and Pattabhi Ramacharyulu [9] studied Prey-predator ecological models with a partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Krishna Gandhi [10] and by Bhaskara Rama Sarma and Pattabhi Ramacharyulu [11], while Ravindra Reddy [12] investigated mutualism between two species. Recently Phani Kumar [13] studied some mathematical models of ecological commensalism. More recently the criteria for a four species syn eco-system was discussed at length by the present authors [14-25].

A Schematic Sketch of the system under investigation is shown here under Fig.1.

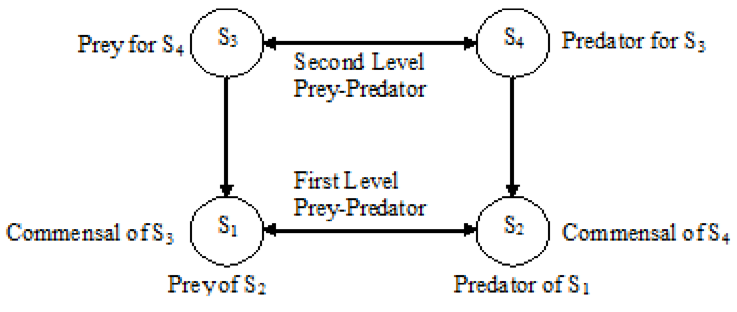

Fig. 1: Schematic Sketch of the Syn Eco - System

2. BASIC EQUATIONS:

The model equations for a four species syn eco-system is given by the following system of first order non-linear ordinary differential equations employing the following notation:

Notation:

S_1	:	Prey for S_2 and commensal for S_3.
S_2	:	Predator surviving upon S_1 and commonsal for S_4.
S_3	:	Host for the commonsal (S_1) and Prey for S_4.
S_4	:	Host of the commonsal (S_2) and Predator surviving upon S_4.
$N_i(t)$	:	The Population strength of S_i at time t, i = 1, 2, 3, 4
t	:	Time instant
a_i	:	Natural growth rate of S_i, i = 1, 2, 3, 4
a_{ii}	:	Self inhibition coefficient of S_i, i = 1, 2, 3, 4
a_{12}, a_{21}	:	Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1
a_{34}, a_{43}	:	Interaction (Prey-Predator) coefficients of S_3 due to S_4 and S_4 due to S_3
a_{13}, a_{24}	:	Coefficients for commensal for S_1 due to the Host S_3 and S_2 due to the Host S_4
$K_i = \frac{a_i}{a_{ii}}$	:	Carrying capacities of S_i, i = 1, 2, 3, 4

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24}, a_{34}, a_{43} are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 - a_{12}N_1N_2 + a_{13}N_1N_3 \tag{2.1}$$

$$\frac{dN_2}{dt} = a_2N_2 - a_2\,N_{22}^{\;2} + a_2\,N_{11}N_2 + a_2\,N_{42}N_4 \tag{2.2}$$

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 - a_{34}N_3N_4 \tag{2.3}$$

$$\frac{dN_4}{dt} = a_4N_4 - a_{44}N_4^2 + a_{43}N_3N_4 \tag{2.4}$$

3. EQUILIBRIUM STATES:

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0,\ i = 1, 2, 3, 4 \tag{3.1}$$

as given in the following Table-1.

Table-1

S.No.	Equilibrium State	Equilibrium Point
1	Fully Washed out state	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
2*	**Only the Host (S_4)of S_2 survives**	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
3*	**Only the Host (S_3)of S_1 survives**	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
4	Only the Predator (S_2) survives	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = 0$

5	Only the Prey (S_1) survives	$\overline{N_1}=\frac{a_1}{a_{11}},\overline{N_2}=0,\overline{N_3}=0,\overline{N_4}=0$
6*	**Prey (S_1) and Predator (S_2) washed out**	$\overline{N_1}=0,\overline{N_2}=0,\overline{N_3}=\frac{\alpha}{\beta},\overline{N_4}=\frac{\gamma}{\beta}$ where $\alpha=a_3a_{44}-a_4a_{34},\ \beta=a_{33}a_{44}+a_{34}a_{43}>0$ $\gamma=a_3a_{43}+a_4a_{33}>0$
7	Prey (S_1) and Host (S_3) of S_1 washed out	$\overline{N_1}=0,\overline{N_2}=\frac{\delta_1}{a_{22}a_{44}},\overline{N_3}=0,\overline{N_4}=\frac{a_4}{a_{44}}$ where $\delta_1=a_2a_{44}+a_4a_{24}>0$
8	Prey (S_1) and Host (S_4) of S_2 washed out	$\overline{N_1}=0,\overline{N_2}=\frac{a_2}{a_{22}},\overline{N_3}=\frac{a_3}{a_{33}},\overline{N_4}=0$
9	Predator (S_2) and Host (S_3) of S_1 washed out	$\overline{N_1}=\frac{a_1}{a_{11}},\overline{N_2}=0,\overline{N_3}=0,\overline{N_4}=\frac{a_4}{a_{44}}$
10	Predator (S_2) and Host (S_4) of S_2 washed out	$\overline{N_1}=\frac{\delta_2}{a_{11}a_{33}},\overline{N_2}=0,\overline{N_3}=\frac{a_3}{a_{33}},\overline{N_4}=0$ where $\delta_2=a_1a_{33}+a_3a_{13}>0$
11	Prey (S_1) and Predator (S_2)survives	$\overline{N_1}=\frac{\alpha_1}{\beta_1},\overline{N_2}=\frac{\gamma_1}{\beta_1},\overline{N_3}=0,\overline{N_4}=0$ where $\alpha_1=a_1a_{22}-a_2a_{12},\ \beta_1=a_{11}a_{22}+a_{12}a_{21}>0$ $\gamma_1=a_1a_{21}+a_2a_{11}>0$
12	Only the Prey (S_1) washed out	$\overline{N_1}=0,\overline{N_2}=\frac{a_2\beta+a_{24}\gamma}{a_{22}\beta},\overline{N_3}=\frac{\alpha}{\beta},\overline{N_4}=\frac{\gamma}{\beta}$
13	Only the predator (S_2) washed out	$\overline{N_1}=\frac{a_1\beta+a_{13}\alpha}{a_{11}\beta},\overline{N_2}=0,\overline{N_3}=\frac{\alpha}{\beta},\overline{N_4}=\frac{\gamma}{\beta}$
14	Only the Host (S_3) of S_1 washed out	$\overline{N_1}=\frac{a_1a_{22}a_{44}-a_{12}\delta_1}{a_{44}\beta_1},\overline{N_2}=\frac{a_1a_{21}a_{44}+a_{11}\delta_1}{a_{44}\beta_1},$ $\overline{N_3}=0,\overline{N_4}=\frac{a_4}{a_{44}}$
15	Only the Host (S_4) of S_2 washed out	$\overline{N_1}=\frac{a_{22}\delta_2-a_2a_{12}a_{33}}{a_{33}\beta_1},\overline{N_2}=\frac{a_{21}\delta_2+a_2a_{11}a_{33}}{a_{33}\beta_1},$ $\overline{N_3}=\frac{a_3}{a_{33}},\overline{N_4}=0$
16	The co-existent state (or) Normal steady state	$\overline{N_1}=\frac{a_{22}\alpha_2-a_{12}\gamma_2}{\beta_1},\overline{N_2}=\frac{a_{11}\gamma_2+a_{21}\alpha_2}{\beta_1},$ $\overline{N_3}=\frac{\alpha}{\beta},\overline{N_4}=\frac{\gamma}{\beta}$ *where* $\alpha_2=a_1+a_{13}\frac{\alpha}{\beta},\ \gamma_2=a_2+a_{24}\frac{\gamma}{\beta}>0$

The present paper deals with the 1st level Prey-Predator washed out states only (Sr. Nos. 2, 3, 6 marked * in the above Table -1). The stability of the other equilibrium states were published several National and International Journals.

4. STABILITY OF THE EQUILIBRIUM STATES:

Let $N = (N_1, N_2, N_3, N_4) = \bar{N} + U$ (4.1)

where $U = (u_1, u_2, u_3, u_4)$ is a perturbation over the equilibrium state $\bar{N} = (\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4)$.

The basic equations (2.1), (2.2), (2.3), (2.4) are quasi-linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt} = AU \tag{4.2}$$

where

$$A = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 - a_{12}\bar{N}_2 + a_{13}\bar{N}_3 & -a_{12}\bar{N}_1 & a_{13}\bar{N}_1 & 0 \\ a_{21}\bar{N}_1 & a_2 - 2a_{22}\bar{N}_2 + a_{21}\bar{N}_1 + a_{24}\bar{N}_4 & 0 & a_{24}\bar{N}_2 \\ 0 & 0 & a_3 - 2a_{33}\bar{N}_3 - a_{34}\bar{N}_3 & -a_{34}\bar{N}_3 \\ 0 & 0 & a_{34}\bar{N}_4 & a_4 - 2a_{44}\bar{N}_4 + a_{43}\bar{N}_3 \end{bmatrix} \tag{4.3}$$

The characteristic equation for the system is $\det[A - \lambda I] = 0$ (4.4)

The equilibrium state is stable, if all the four roots of the equation (4.4) are negative, in case they are real or have negative real parts, in case they are complex.

5. STABILITY OF THE 1ST LEVEL PREY-PREDATOR WASHED OUT EQUILIBRIUM STATES: (SL. NOS. 2,3,6 MARKED * IN TABLE .1)

5.1 Equilibrium point $\bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = \frac{a_4}{a_{44}}$:

The corresponding linearized equations for the perturbations u_1, u_2, u_3, u_4 are

$$\frac{du_1}{dt} = a_1 u_1, \frac{du_2}{dt} = p_2 u_2 \tag{5.1.1}$$

$$\frac{du_3}{dt} = p_3 u_3, \quad \frac{du_4}{dt} = k_3 a_{43} u_3 - a_4 u_4 \tag{5.1.2}$$

Here $p_2 = a_2 + k_4 a_{24} > 0, \quad p_3 = a_3 - k_4 a_{34}$ (5.1.3)

The characteristic equation for which is

$$(\lambda_1 - a_1)(\lambda - p_2)(\lambda - p_3)(\lambda + a_4) = 0 \tag{5.1.4}$$

Two of the four roots a_1, p_2 are positive and $-a_4$ is negative. Hence the state is **unstable.**

Case (A): If $p_3 > 0$ (ie, $a_3 > k_4\ a_{34}$)

The solutions of the equations (5.1.1), (5.1.2) are

$$u_1 = u_{10} e^{a_1 t}, \quad u_2 = u_{20} e^{P_2 t} \tag{5.1.5}$$

$$u_3 = u_{30} e^{P_3 t}, \quad u_4 = (u_{40} - P) e^{-a_4 t} + P e^{P_3 t} \tag{5.1.6}$$

Here $P = \frac{k_4 a_{43} u_{30}}{p_3 + a_4}$ (5.1.7)

and $u_{10}, u_{20}, u_{30}, u_{40}$ are the initial values of $u_1, u_2\ u_3, u_4$ respectively.

In the three equilibrium states, there would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations. The solution curves are illustrated in Figures (2) to (4) and the conclusions are presented here.

Case (i): If $u_{10} < u_{30} < u_{40} < u_{20}$ and $a_1 < a_4 < p_3 < p_2$

In this case the natural birth rates of the Prey (S_1), Host (S_4) of S_2, Host (S_3) of S_1 and the Predator (S_2) are in ascending order. Initially the Host (S_4) of S_2 dominates over the Host (S_3) of S_1 till the time instant t^*_{34} and thereafter the dominance is reversed. The time t^*_{34} may be called the dominance time of S_4 over S_3.

Here $$t^*_{34} = \frac{1}{p_3 + a_4} \log\left(\frac{u_{40} - P}{u_{30} - P}\right) \tag{5.1.8}$$

Case (ii): If $u_{30} < u_{10} < u_{20} < u_{40}$ and $a_1 < p_2 < a_4 < p_3$

In this case the natural birth rates of the Prey (S_1), Predator (S_2), Host (S_4) of S_2 and the Host (S_3) of S_1 are in ascending order. Initially the Host (S_4) of S_2, Predator (S_2), Prey (S_1) dominates over the Host (S_3) of S_1 till the time instant t^*_{34}, t^*_{32}, t^*_{31} respectively and thereafter the dominance is reversed.

Here $$t^*_{32} = \frac{1}{p_2 - p_3} \log\left(\frac{u_{30}}{u_{20}}\right), t^*_{31} = \frac{1}{a_1 - p_3} \log\left(\frac{u_{30}}{u_{10}}\right) \tag{5.1.9}$$

Case (iii): If $u_{40} < u_{10} < u_{20} < u_{30}$ and $p_2 < a_1 < p_3 < a_4$

In this case the natural birth rates of the Predator (S_2), Prey (S_1), Host (S_3) of S_1 and the Host (S_4) of S_2 are in ascending order. Initially the Host (S_3) of S_1, Predator (S_2), Prey (S_1) dominates over the Host (S_4) of S_2 till the time instant t^*_{43}, t^*_{42}, t^*_{41} respectively and thereafter the dominance is reversed. Also the Predator (S_2) dominates over the Prey (S_1) till the time instant t^*_{12} and the dominance gets reversed thereafter.

Here $$t^*_{12} = \frac{1}{a_1 - p_2} \log\left(\frac{u_{20}}{u_{10}}\right) \tag{5.1.10}$$

Case (B): If $p_3 < 0$ (ie, $a_3 < k_4\, a_{34}$)

The solutions in this case are some as in case (A) and the solution curves are illustrated in Figures (5) to (8).

Case (i): If $u_{10} < u_{30} < u_{20} < u_{40}$ and $a_1 < p_2 < p_3 < a_4$

In this case the natural birth rates of the Host (S_3) of S_1, Host (S_4) of S_2, Prey (S_1) and the Predator (S_2) are in ascending order. Initially the Host (S_4) of S_2 dominates over the Predator (S_2), Prey (S_1) till the time instant t^*_{24}, t^*_{14} respectively and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Prey (S_1) till the time instant t^*_{13} and the dominance gets reversed thereafter.

Case (ii): If $u_{20} < u_{40} < u_{30} < u_{10}$ and $p_2 < p_3 < a_4 < a_1$

In this case the natural birth rates of the Host (S_3) of S_1, Host (S_4) of S_2, Predator (S_2) and the Prey (S_1) are in ascending order. Initially the Host (S_3) of S_1, Host (S_4) of S_2 dominates over the Predator (S_2) till the time instant t_{23}^*, t_{24}^* respectively and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Host (S_4) of S_2 till the time instant t_{43}^* and the dominance gets reversed thereafter.

Case (iii): If $u_{30} < u_{20} < u_{40} < u_{10}$ and $p_3 < a_1 < p_2 < a_4$

In this case the natural birth rates are same as in case (i). Initially the Prey (S_1), Host (S_4) of S_2 dominates over the Predator (S_2) till the time instant t_{21}^*, t_{24}^* respectively and thereafter the dominance is reversed.

Case (iv): If $u_{40} < u_{30} < u_{20} < u_{10}$ and $p_2 < a_1 < a_4 < p_3$

In this case the Host (S_4) of S_2 has the least natural birth rate and the Prey (S_1) dominates the Predator (S_2), Host (S_3) of S_1, Host (S_4) of S_2 in natural growth rate as well as in its population strength.

5.1.A. Trajectories of Perturbations:

The trajectories in the $u_1 - u_2$ plane given by $x^{p_2} = y_1^{a_1}$ (5.1.11)

and are shown in Fig.9 and the trajectories in the other planes are

$$x^{p_3} = y_2^{a_1},\ y_1^{p_3} = y_2^{p_2},\ y_3 = (1-A)\,x^{\frac{-a_4}{a_1}} + A\,x^{\frac{p_3}{a_1}}, \tag{5.1.12}$$

$$y_3 = (1-A)\,y_1^{\frac{-a_4}{p_2}} + Ay_1^{\frac{p_3}{p_2}},\ y_3 = (1-A)\,y_2^{\frac{-a_4}{p_3}} + ay_2 \tag{5.1.13}$$

where $x = \dfrac{u_1}{u_{10}},\ y_1 = \dfrac{u_2}{u_{20}},\ y_2 = \dfrac{u_3}{u_{30}},\ y_3 = \dfrac{u_4}{u_{40}},\ A = \dfrac{P}{u_{40}}$ (5.1.14)

5.2. Equilibrium point $\overline{N}_1 = 0,\ \overline{N}_2 = 0,\ \overline{N}_3 = \dfrac{a_3}{a_{33}},\ \overline{N}_4 = 0$:

The corresponding linearized equations for the perturbations u_1, u_2, u_3, u_4 are

$$\frac{du_1}{dt} = p_1u_1,\qquad \frac{du_2}{dt} = a_2u_2 \tag{5.2.1}$$

$$\frac{du_3}{dt} = -a_3u_3 - k_3a_3\ u_{44},\ \frac{du_4}{dt} = a_4u_4 \tag{5.2.2}$$

Here $p_1 = a_1 + k_3a_{13} > 0$ (5.2.3)

The characteristic equation for which is

$$(\lambda - p_1)(\lambda - a_2)(\lambda + a_3)(\lambda - a_4) = 0 \tag{5.2.4}$$

The roots p_1, a_2, a_4 are positive and $-a_3$ is negative. Hence the state is **unstable** and the solutions of the equations (5.2.1), (5.2.2) are

$$u_1 = u_{10}\ e^{p_1t},\quad u_2 = u_{20}\ e^{a_2t} \tag{5.2.5}$$

$$u_3 = (u_{30} + Q)\,e^{-a_3t} - Qe^{a_4t},\ u_4 = u_{40}e^{a_4t} \tag{5.2.6}$$

where $Q = \dfrac{k_3a_{34}u_{40}}{a_3 + a_4}$ (5.2.7)

The solution curves are illustrated in Figures (10) to (12) and the conclusions are presented here.

Case (i): If $u_{10} < u_{20} < u_{40} < u_{30}$ and $a_2 < p_1 < a_3 < a_4$

In this case the natural birth rates of the Predator (S_2), Prey (S_1), Host (S_3) of S_1 and the Host (S_4) of S_2 are in ascending order. Initially the Predator (S_2) dominates over the Prey (S_1) till the time instant t_{12}^* and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Host (S_4) of S_2 till the time instant t_{43}^* and the dominance gets reversed thereafter.

Here $t_{12}^* = \frac{1}{p_1 - a_2} \log\left(\frac{u_{20}}{u_{10}}\right)$, $t_{43}^* = \frac{1}{a_3 + a_4} \log\left(\frac{u_{30} + Q}{u_{40} + Q}\right)$ (5.2.8)

Case (ii): If $u_{30} < u_{10} < u_{40} < u_{20}$ and $p_1 < a_2 < a_3 < a_4$

In this case the natural birth rates of the Prey (S_1), Predator (S_2), Host (S_3) of S_1 and the Host (S_4) of S_2 are in ascending order. Initially the Predator (S_2) dominates over the Host (S_3) of S_1, Host (S_4) of S_2 till the time instant t_{32}^*, t_{42}^* respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates its Host (S_3) till the time instant t_{31}^* and the dominance gets reversed thereafter.

Here $t_{42}^* = \frac{1}{a_2 - a_4} \log\left(\frac{u_{40}}{u_{20}}\right)$ (5.2.9)

Case (iii): If $u_{40} < u_{20} < u_{30} < u_{10}$ and $a_3 < p_1 < a_4 < a_2$

In this case the natural birth rates of the Host (S_3) of S_1, Prey (S_1), Host (S_4) of S_2 and the Predator (S_2) are in ascending order. Initially the Prey (S_1) dominates over the Predator (S_2), Host (S_4) of S_2 till the time instant t_{21}^*, t_{41}^* respectively and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Predator (S_2), Host (S_4) of S_2 till the time instant t_{23}^* t_{43}^* respectively and the dominance gets reversed thereafter.

Here $t_{41}^* = \frac{1}{p_1 - a_4} \log\left(\frac{u_{40}}{u_{10}}\right)$ (5.2.10)

5.2.A. Trajectories of Perturbations:

The trajectories in the $u_1 - u_2$ plane gives by $x^{a_2} = y_1^{P_1}$ (5.2.11)
and are shown in Fig.13 and the trajectories in the other planes are

$$x^{a_4} = y_3^{P_1},\ y_1^{a_4} = y_3^{a_2},\ y_2 = (1+B)\, x^{\frac{-a_3}{P_1}} - Bx^{\frac{a_4}{P_1}} \quad (5.2.12)$$

$$y_2 = (1+B)\, y_1^{\frac{-a_3}{a_2}} - By_1^{\frac{a_4}{a_2}},\ y_2 = (1+B)\, y_3^{\frac{-a_3}{a_4}} - By_3 \quad (5.2.13)$$

where $B = \frac{P}{u_{30}}$ (5.2.14)

5.3 Equilibrium point $\overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = \frac{\alpha}{\beta}, \overline{N}_4 = \frac{\gamma}{\beta}$:

Here $\alpha = a_3 a_{44} - a_4 a_{34}$, $\beta = a_{33} a_{44} + a_{34} a_{43} > 0$ (5.3.1)

and $\gamma = a_3 a_{43} + a_4 a_{33} > 0$ (5.3.2)

The corresponding linearized equations for the perturbations u_1, u_2, u_3, u_4 are

$$\frac{du_1}{dt} = q_1 u_1,\ \frac{du_2}{dt} = q_2 u_2 \quad (5.3.3)$$

$$\frac{du_3}{dt} = q_3 u_3 - a_{34} \frac{\alpha}{\beta} u_4,\ \frac{du_4}{dt} = a_{43} \frac{\gamma}{\beta} u_3 + q_4 u_4 \quad (5.3.4)$$

Here $$q_1 = a_1 + a_{13}\frac{\alpha}{\beta},\ q_2 = a_2 + a_{24}\frac{\gamma}{\beta} > 0 \tag{5.3.5}$$

$$q_3 = a_3 - 2a_{33}\frac{\alpha}{\beta} - a_{34}\frac{\gamma}{\beta},\ q_4 = a_4 - 2a_{44}\frac{\gamma}{\beta} + a_{43}\frac{\alpha}{\beta} \tag{5.3.6}$$

The characteristic equation for which is

$$(\lambda - q_1)(\lambda - q_2)\left[\lambda^2 - (q_3 + q_4)\lambda + \left(q_3 q_4 + a_{34}a_{43}\frac{\alpha\gamma}{\beta^2}\right)\right] = 0 \tag{5.3.7}$$

One of the four roots q_2 is positive. Hence the state is **unstable**. Let λ_1, λ_2 be the zeros of the quadratic polynomial on the R.H.S of the equation (5.3.7)

Case (A): When $\alpha > 0$ and $(q_3 - q_4)^2 > 4a_{34}\ a_{43}\frac{\alpha\gamma}{\beta^2}$

The roots q_1, λ_1 are positive and λ_2 is negative and the solutions of the equations (5.3.3), (5.3.4) are

$$u_1 = u_{10}\ e^{q_1 t},\ u_2 = u_{20}\ e^{q_2 t} \tag{5.3.8}$$

$$u_3 = \left[\frac{a_{34}\ \alpha u_{40} + \beta(q_3 - \lambda_2)u_{30}}{\beta(\lambda_1 - \lambda_2)}\right] e^{\lambda_1 t} + \left[\frac{a_{34}\ \alpha u_{40} + \beta(q_3 - \lambda_1)u_{30}}{\beta(\lambda_2 - \lambda_1)}\right] e^{\lambda_2 t} \tag{5.3.9}$$

$$u_4 = \left[\frac{a_{34}\ \alpha u_{40} + \beta(q_3 - \lambda_2)u_{30}}{a_{34}\alpha(\lambda_1 - \lambda_2)}(q_3 - \lambda_1)\right] e^{\lambda_1 t} + \left[\frac{a_{34}\ \alpha u_{40} + \beta(q_3 - \lambda_1)u_{30}}{a_{34}\alpha(\lambda_2 - \lambda_1)}(q_3 - \lambda_2)\right] e^{\lambda_2 t} \tag{5.3.10}$$

The solution curves are illustrated in Figures (14), (15) and conclusions are presented here.

Case (i): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_4 < q_2 < a_3 < q_1$

In this case the natural birth rates of the Host (S_4) of S_2, Predator (S_2), Host (S_3) of S_1 and the Prey (S_1) are in ascending order. Initially the Host (S_4) of S_2 dominates over the Host (S_3) of S_1, Predator (S_2) till the time instant t^*_{34}, t^*_{24} respectively and thereafter the dominance is reversed.

Case (ii): If $u_{30} < u_{20} < u_{40} < u_{10}$ and $a_3 < q_2 < a_4 < q_1$

In this case the Predator (S_2) has the least natural birth rate and the Prey (S_1) dominates the Host (S_4) of S_2, Host (S_3) of S_1, Predator (S_2) in natural growth rate as well as in its population strength.

Case (B): when $\alpha > 0$ and $(q_3 - q_4)^2 < 4a_{34}\ a_{43}\frac{\alpha\gamma}{\beta^2}$

The roots q_1, q_2 are positive and λ_1, λ_2 are complex. The solutions in this case are same as in case (A) and this is illustrated in Fig. 16.

Case (C): When $\alpha < 0, \left[(q_3 - q_4)^2 - 4a_{34}\ a_{43}\frac{\alpha\gamma}{\beta^2}\right] > 0$

The roots q_1, λ_2 are negative and λ_1 is positive. The solutions in this case are same as in case (A) and the solution curves are illustrated in figures (17) to (20) and the conclusions are presented here.

Case (i): If $u_{10} < u_{20} < u_{40} < u_{30}$ and $q_1 < q_2 < a_3 < a_4$

In this case the natural birth rates of the Prey (S_1), Predator (S_2), Host (S_3) of S_1 and the (S_4) of S_2 are in ascending order. Initially the Host (S_3) of S_1 dominates over the Host (S_4) of S_2 till the time instant t^*_{43} and thereafter the dominance is reversed.

Here $t_{43}^{*} = \frac{1}{\lambda_1 - \lambda_2} \log\left[\frac{(\beta b_7 - b_1 b_3)u_{30} + (\beta b_6 - b_1^2)u_{40}}{(b_1 b_2 - \beta b_5)u_{30} + (b_1^2 - \beta b_4)u_{40}}\right]$ (5.3.11)

where $b_1 = a_{34}\alpha,\ b_2 = \beta(q_3 - \lambda_2),\ b_3 = \beta(q_3 - \lambda_1)$ (5.3.12)

$b_4 = b_1(q_3 - \lambda_1), b_5 = b_2(q_3 - \lambda_1), b_6 = b_1(q_3 - \lambda_2), b_7 = b_3(q_3 - \lambda_2)$ (5.3.13)

Case (ii): If $u_{20} < u_{40} < u_{10} < u_{30}$ and $q_2 < q_1 < a_4 < a_3$

In this case the natural birth rates of the Prey (S_1), Predator (S_2), Host (S_4) of S_2 and the Host (S_3) of S_1 are in ascending order. Initially the Prey (S_1) dominates over the Predator (S_2) and its Host (S_4) till the time instant t_{21}^{*} and t_{41}^{*} respectively and thereafter the dominance is reversed.

Here $t_{21}^{*} = \frac{1}{q_1 - q_2} \log\left(\frac{u_{20}}{u_{10}}\right)$ (5.3.14)

Case (iii): If $u_{30} < u_{10} < u_{40} < u_{20}$ and $q_2 < q_1 < a_3 < a_4$

In this case the natural birth rates are same as in case (i). Initially the Predator (S_2) dominates over the Host (S_3) of S_1, Host (S_4) of S_2 till the time instant t_{32}^{*}, t_{42}^{*} respectively and thereafter the dominance is reversed.

Case (iv): If $u_{40} < u_{30} < u_{20} < u_{10}$ and $a_3 < a_4 < q_1 < q_2$

In this case the natural birth rates of the Prey (S_1), Host (S_3) of S_1, Host (S_4) of S_2 and the Predator (S_2) are in ascending order. Initially the Prey (S_1) dominates over the Predator (S_2), Host (S_3) of S_1, Host (S_4) of S_2 till the time instant $t_{21}^{*}, t_{31}^{*}, t_{41}^{*}$ respectively and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Host (S_4) of S_2 till the time instant t_{43}^{*} and the dominance gets reversed thereafter.

5.3. A. Trajectories of Perturbations:

The trajectories in the $u_1 - u_2$ plane given by $x^{q_2} = y_1^{q_1}$ (5.3.15)
and are shown in Fig.21 and the trajectories in the other planes are

$$y_2 = A_1 x^{\frac{\lambda_1}{q_1}} + B_1 x^{\frac{\lambda_2}{q_1}},\ y_3 = A_2 x^{\frac{\lambda_1}{q_1}} + B_2 x^{\frac{\lambda_2}{q_1}} \tag{5.3.16}$$

$$y_2 = A_1 y_1^{\frac{\lambda_1}{q_2}} + B_1 y_1^{\frac{\lambda_2}{q_2}},\ y_3 = A_2 y_1^{\frac{\lambda_1}{q_2}} + B_2 y_1^{\frac{\lambda_2}{q_1}} \tag{5.3.17}$$

where $$A_1 = \frac{a_{34}\alpha u_{40} + \beta(q_3 - \lambda_2)u_{30}}{\beta(\lambda_1 - \lambda_2)u_{30}},\ B_1 = \frac{a_{34}\alpha u_{40} + \beta(q_3 - \lambda_1)u_{30}}{\beta(\lambda_2 - \lambda_1)u_{30}} \tag{5.3.18}$$

$$A_2 = \frac{a_{34}\alpha u_{40} + \beta(q_3 - \lambda_2)u_{30}}{a_{34}\alpha(\lambda_1 - \lambda_2)u_{30}}(q_3 - \lambda_1) \tag{5.3.19}$$

$$B_2 = \frac{a_{34}\alpha u_{40} + \beta(q_3 - \lambda_1)u_{30}}{a_{34}\alpha(\lambda_2 - \lambda_1)u_{30}}(q_3 - \lambda_2) \tag{5.3.20}$$

6. PERTURBATION GRAPHS.

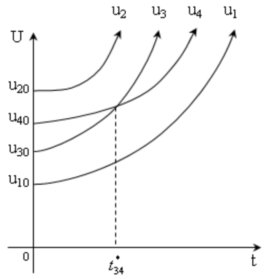

Fig. 2

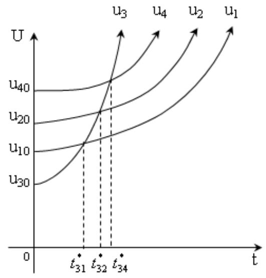

Fig. 3

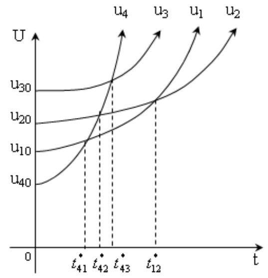

Fig. 4

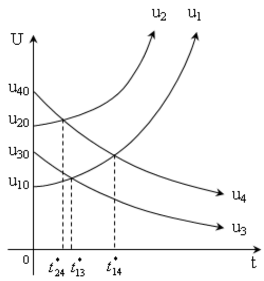

Fig. 5

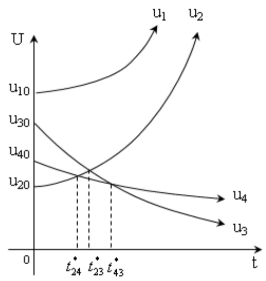

Fig. 6

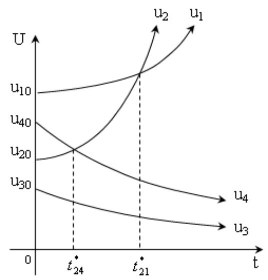

Fig. 7

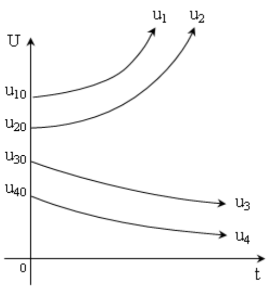

Fig. 8

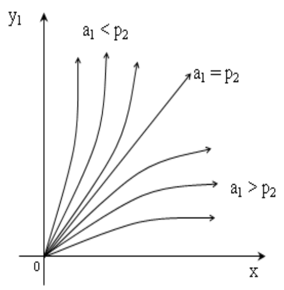

Fig. 9

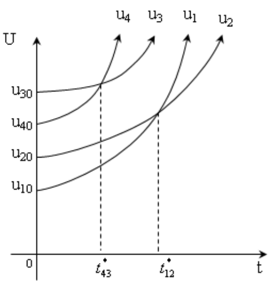

Fig. 10

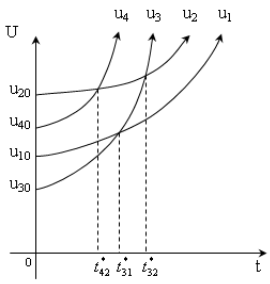

Fig. 11

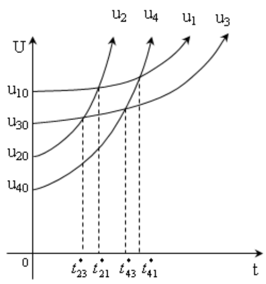

Fig. 12

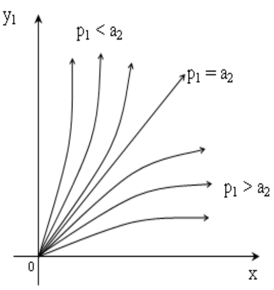

Fig. 13

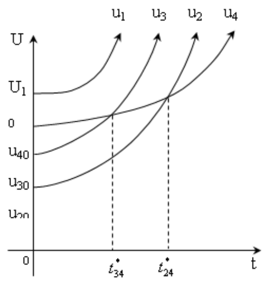

Fig. 14

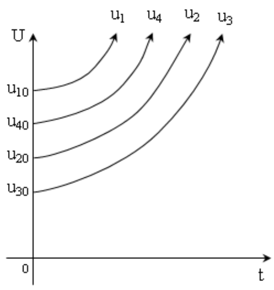

Fig. 15

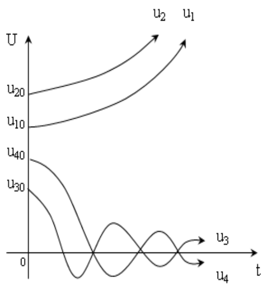

Fig. 16

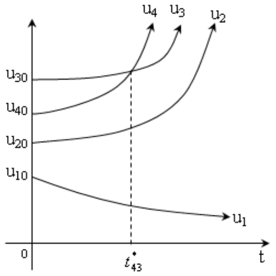

Fig. 17

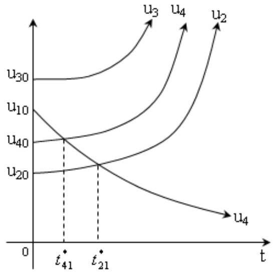

Fig. 18

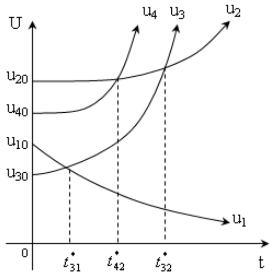

Fig. 19

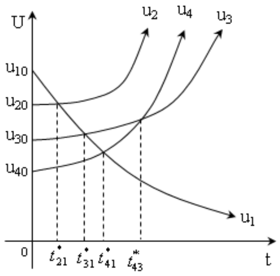

Fig. 20

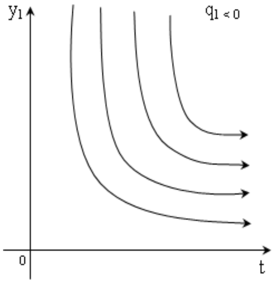

Fig. 21

7. NUMERICAL APPROACH OF THE GROWTH RATE EQUATIONS

The numerical solutions of the growth rate equations (2.1), (2.2), (2.3) and (2.4) computed employing the fourth order Runge - Kutta method for specific values of the various parameters that characterize the model and the initial conditions. For this Mat Lab has been used and the results are illustrated in Figures (22) to (25).

Consider the model parameter values
a_1=0.7, a_2=1.2, a_3=2.8, a_4=0.48, a_{12}=0.87,a_{13}=0.43,a_{21}=0.32, a_{24}=0.18, a_{34}=1.18, a_{43}=0.18, K_1=3.5, K_2=4.8, K_3=5.6, K_4=1.2

Case (a): If $N_{i0} < \dfrac{K_i}{2}$, i = 1, 2, 3, 4.

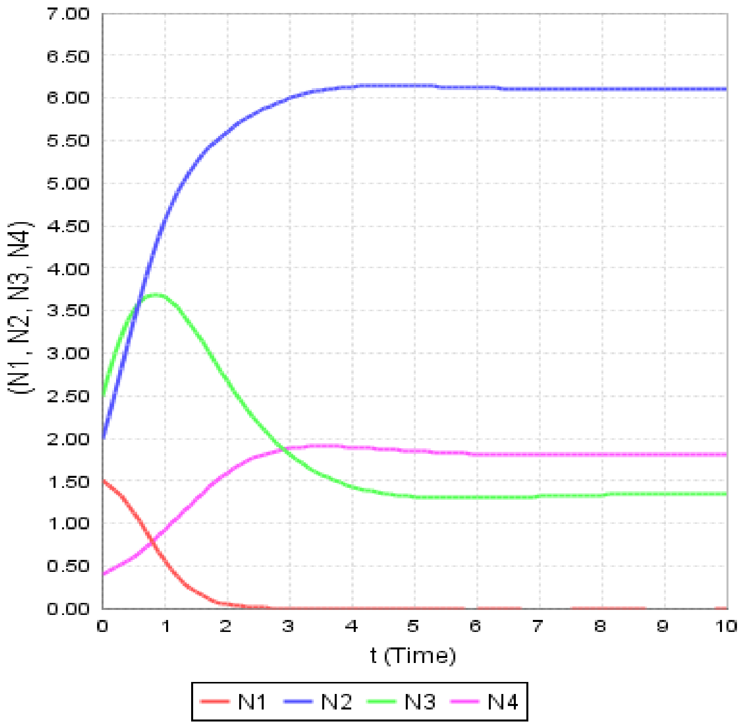

Figure 22: Variation of N_1, N_2, N_3 and N_4 against time(t) for N_{10}=1.5,N_{20}=2, N_{30}=2.5, N_{40}=0.4

Case (b): If $N_{i0} > K_i$, i = 1, 2, 3, 4.

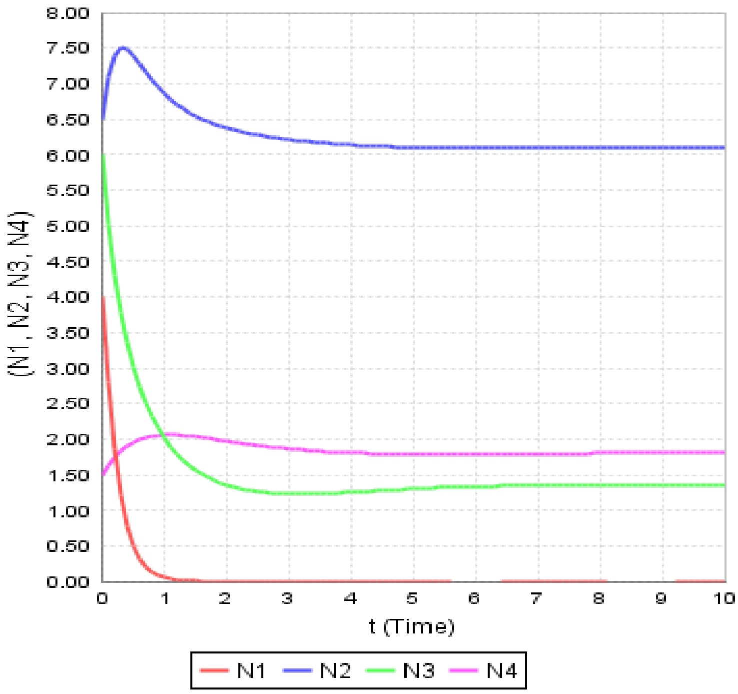

Figure 23: Variation of N_1, N_2, N_3 and N_4 against time(t) for N_{10}=4,N_{20}=6.5, N_{30}=6, N_{40}=1.5

Case (c): If $\frac{K_i}{2} < N_{i0} < K_i$, i = 1, 2, 3, 4.

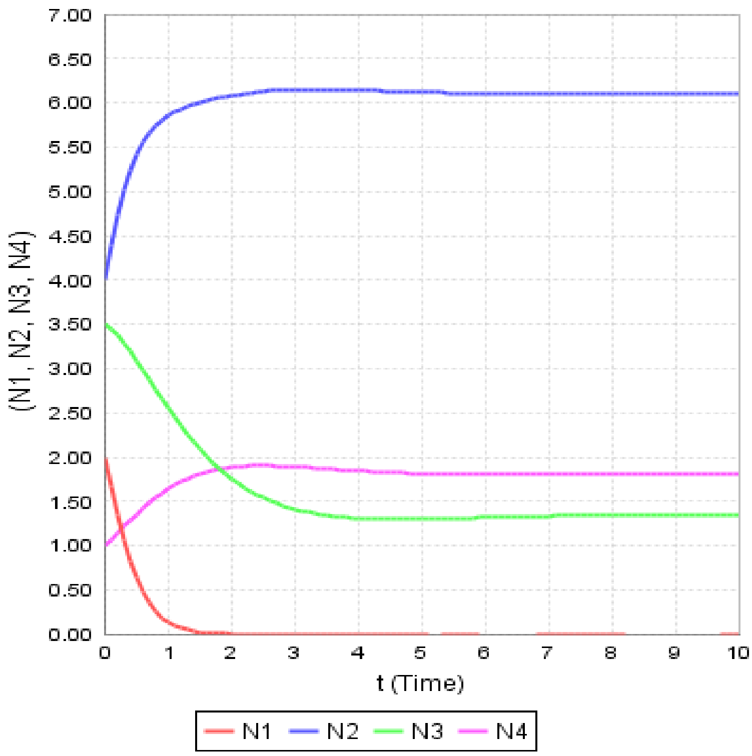

Figure 24: Variation of N_1, N_2, N_3 and N_4 against time(t) for N_{10}=2,N_{20}=4, N_{30}=3.5, N_{40}=1

Case (d): If $N_{i0} = K_i$, i = 1, 2, 3, 4.

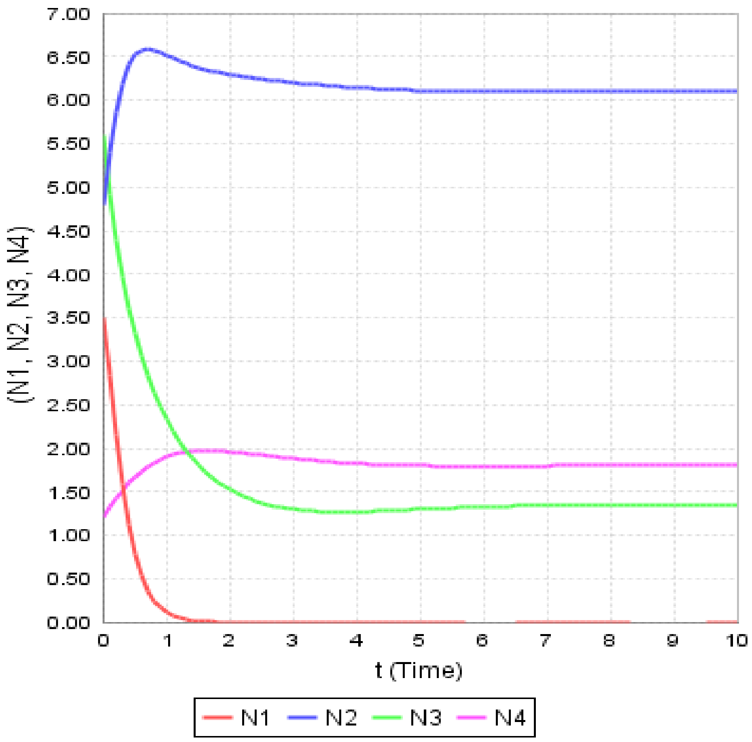

Figure 25: Variation of N_1, N_2, N_3 and N_4 against time(t) for N_{10}=3.5,N_{20}=4.8, N_{30}=5.6, N_{40}=1.2

8. OPEN PROBLEM:

Investigate some relation-chains between the species such as Prey-Predation, Neutralism, Commensalism, Mutualism, Competition and Ammensalism between four species (S_1, S_2, S_3, S_4) with the population relations.

S_1 a Prey to S_2 and Commensal to S_3, S_2 is a Predator living on S_1 and Commensal to S_4, S_3 a Host to S_1, S_4 a Host to S_2 and S_3 a Prey to S_4, S_4 a Predator to S_3.

The present paper deals with the study on stability of 1st level Prey-Predator washed out states only of the above problem. The stability of the other equilibrium states were published several National and International Journals.

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Source of support: Nil, Conflict of interest: None Declared

