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A STRONG FORM OF COMPACT SPACES

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ABSTRACT

In this paper, we introduce and study the notion of $\mathcal{N}$-λ-open sets as a generalization of λ-open sets.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper (X,τ) and (Y,σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Maki [3] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel, that is, to the intersection of all open super sets of A. Arenas et.al. [1] Introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets. Let A be a subset of a topological space (X,τ). The closure and the interior of a set A is denoted by Cl (A), Int (A) respectively. A subset A of a topological space (X,τ) is said to be λ-closed [1] if $A = B \cap C$, where B is a Λ-set and C is a closed set of X. The complement of λ-closed set is called λ-open [1]. A point $x \in X$ in a topological space (X,τ) is said to be λ-cluster point of A [2] if for every λ-open set U of X containing x, $A \cap U \neq \phi$. The set of all λ-cluster points of A is called the λ-closure of A and is denoted by $Cl_\lambda(A)$ [2]. A point $x \in X$ is said to be the λ-interior point of A if there exists a λ-open set U of X containing x such that $U \subset A$. The set of all λ-interior points of A is said to be the λ-interior of A and is denoted by $Int_\lambda(A)$. A set A is λ-closed (resp.λ-open) if and only if $Cl_\lambda(A) = A$ (resp. $Int_\lambda(A) = A$) [2]. The family of all λ-open (resp. λ-closed) sets of X is denoted by $\lambda O(X)$ (resp. $\lambda C(X)$). The family of all λ-open (resp. λ-closed) sets of a space (X,τ) containing the point $x \in X$ is denoted by $\lambda O(X, x)$ (resp. $\lambda C(X, x)$).

2. $\mathcal{N}$-λ-OPEN SETS

Definition 2.1: A subset A of a topological space X is said to be $\mathcal{N}$-λ-open if for every $x \in A$, there exists a λ-open subset $U_x \in X$ containing x such that $U_x - A$ is finite set. The complement of an $\mathcal{N}$-λ-open subset is said to be $\mathcal{N}$-λ-closed.

The family of $\mathcal{N}$-λ-open (resp.$\mathcal{N}$-λ-closed) subsets of a space (X,τ) is denoted by $\mathcal{N}\lambda O(X)$ (resp. $\mathcal{N}\lambda C(X)$).

Proposition 2.2: Every λ-open set is $\mathcal{N}$-λ-open. Converse not true.

Example 2.3: Let $X = \{a,b,c\}$, $\tau = \{\phi, \{a\}, X\}$, then $\{b\}$ is $\mathcal{N}$-open but not $\mathcal{N}$-λ-open (since X is a finite set).

Corollary 2.4: Every open set is $\mathcal{N}$-λ-open, but not conversely.

Proof: Follows from the fact that every open set is λ-open.

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Lemma 2.5: A subset A of a topological space X is $\mathcal{N}$-λ-open if and only if for every $x \in A$, there exists a λ-open subset U_x containing x and a finite subset C such that $U_x - C \subset A$.

Proof: Let A be $\mathcal{N}$-λ-open and $x \in A$, then there exists a λ-open subset U_x containing x such that $U_x - A$ is finite. Let $C = U_x - A = U_x \cap (X - A)$. Then $U_x - C \subseteq A$. Conversely, let $x \in A$. Then there exists a λ-open subset U_x containing x and a finite subset C such that $U_x - C \subseteq A$. Thus, $U_x - C \subseteq A$ and $U_x - A$ is a finite set.

Theorem 2.6: Let X be a topological space and $F \subseteq X$. If F is $\mathcal{N}$-λ-closed, then $F \subseteq K \cup C$ for some λ-closed subset K and a finite subset C.

Proof: If F is $\mathcal{N}$-λ-closed, then $X - F$ is $\mathcal{N}$-λ-open and hence for every $x \in X - F$, there exists a λ-open set U containing x and a finite set C such that $U - C \subseteq X - F$. Thus $F \subset X - (U - C) = X - (U \cap (X - C)) = (X - U) \cup C$. Let $K = X - U$. Then K is a λ-closed set such that $F \subseteq K \cup C$.

Lemma 2.7: The union of any family of $\mathcal{N}$-λ-open sets is $\mathcal{N}$-λ-open.

Proof: If $\{U_i : i \in I\}$ is a collection of $\mathcal{N}$-λ-open subsets of X and $x \in \bigcup_{i=1}^{n} U_i$. Then $x \in U_j$ for some $j \in I$. This implies that there exists a λ-open subset V of X containing x such that $V - U_j$ is finite. Since $V - \bigcup_{i=1}^{n} U_i \subseteq V - U_j$, then $V - \bigcup_{i=1}^{n} U_i$ is finite. Therefore, $\bigcup_{i=1}^{n} U_i \in \mathcal{N}\lambda O(X)$.

The intersection of all $\mathcal{N}$-λ-closed sets of X containing A is called the $\mathcal{N}$-λ-closure of A and is denoted by $\mathcal{N}Cl_\lambda A)$. And the union of all $\mathcal{N}$-λ-open sets of X contained in A is called the $\mathcal{N}$-λ-interior and is denoted by $\mathcal{N}Int_\lambda(A)$.

The proof of the following lemma is obvious and hence omitted.

Lemma 2.8: Let A be a subset of a topological space (X, τ). Then

(i) $x \in \mathcal{N}Cl_\lambda(A)$ if and only if $A \cap U \neq \phi$ for every $U \in \mathcal{N}\lambda O(X, x)$;

(ii) A is $\mathcal{N}$-λ-closed if and only if $A = \mathcal{N}Cl_\lambda(A)$;

(iii) $\mathcal{N}Cl_\lambda(A)$ is $\mathcal{N}$-λ-closed.

Corollary 2.9: The intersection of an $\mathcal{N}$-λ-open set with an open set is $\mathcal{N}$-λ-open.

Question: Does there exist an example for the intersection of $\mathcal{N}$-λ-open sets is $\mathcal{N}$-λ-open?

3. λ - COMPACT SPACES

Definition 3.1: A collection $\{U_\alpha : \alpha \in \Delta\}$ of λ-open sets in a topological space X is called a λ-open cover of a subset B of X if $B \subset \{U_\alpha : \alpha \in \Delta\}$ holds.

Definition 3.2: A topological space X is said to be λ- compact if every λ-open cover of X has a finite subcover. A subset A of a topological space X is said to be λ-compact relative to X if every cover of A by λ-open sets of X has a finite subcover.

Theorem 3.3: If X is a topological space such that every λ-open subset is λ-compact relative to X, then every subset is λ-compact relative to X.

Proof: Let B be an arbitrary subset of X and let $\{U_i : i \in I\}$ be the cover of B by λ-open sets of X. Then the family $\{U_i : i \in I\}$ is a λ-open cover of the λ-open set $\cup\{U_i : i \in I\}$. Hence by hypothesis there is a finite subfamily $\{U_{i_j} : j \in N_0\}$ which covers $\cup\{U_i : i \in I\}$. This subfamily is also a cover of the set B.

Theorem 3.4: A subset A of a topological space is λ-compact relative to X if and only if for any cover $\{V_\alpha : \alpha \in \Delta\}$ of A by $\mathcal{N}$-λ-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{V_\alpha : \alpha \in \Delta_0\}$.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of A and $V_\alpha \in \mathcal{N}\lambda O(X)$ .For each $x \in A$, there exists $\alpha(x) \in \Delta$ such that $x \in V_{\alpha(x)}$. Since $V_{\alpha(x)}$ is $\mathcal{N}$-λ-open, there exist a λ-open set $U_{\alpha(x)}$ such that $x \in U_{\alpha(x)}$ and $U_{\alpha(x)} - V_{\alpha(x)}$ is finite. The family $\{U_{\alpha(x)} : x \in A\}$ is a λ-open cover of A. Since A is λ-compact relative to X, there exists a finite subset $x_1, x_2, \ldots x_n$, such that $A \subseteq \cup\{U_{\alpha(x_i)} : i \in F\}$, where $F = \{1, 2, \ldots n\}$. Now, we have, $A \subseteq \bigcup_{i \in F}\left(\left(U_{\alpha(x_i)} - V_{\alpha(x_i)}\right) \cup V_{\alpha(x_i)}\right) = \bigcup_{i \in F}\left(U_{\alpha(x_i)} - V_{\alpha(x_i)}\right) \cup \bigcup_{i \in F} V_{\alpha(x_i)}$. For each x_i, $U_{\alpha(x_i)} - V_{\alpha(x_i)}$ is a finite set and there exists a finite subset $\Delta(x_i)$ of Δ such that $\left(U_{\alpha(x_i)} - V_{\alpha(x_i)}\right) \cap A \subseteq \cup\{V_\alpha : \alpha \in \Delta(x_i)\}$. Therefore, we have $A \subseteq \left(\bigcup_{i \in F}(\cup\{V_\alpha : \alpha \in \Delta(x_i)\})\right) \cup \left(\bigcup_{i \in F} V_{\alpha(x_i)}\right)$. Hence A is λ-compact relative to X.

Corollary 3.5: For any topological space X, the following properties are equivalent:
(i) X is λ-compact.
(ii) Every $\mathcal{N}$-λ-open cover of X admits a finite subcover.

Theorem 3.6: A topological space X is λ-compact if and only if every proper $\mathcal{N}$-λ-closed set is λ-compact with respect to X.

Proof: Let A be a proper $\mathcal{N}$-λ-closed subset of X. Let $\{U_\alpha : \alpha \in \Delta\}$ be a cover of A by λ-open sets of X. Now for each $x \in X - A$, there is a λ-open set V_x such that $V_x - A$ is finite . Then $\{U_\alpha : \alpha \in \Delta\} \cup \{V_x : x \in X - A\}$ is a λ-open cover of X. Since X is λ-compact, there exist a finite subset Δ_1 of Δ and a finite number of points, say, $x_1, x_2, \ldots x_n$ in $X - A$ such that $X = (\cup\{U_\alpha : \alpha \in \Delta_1\}) \cup (\cup\{V_{x_i} : 1 \le i \le n\})$ hence $A \subset (\cup\{U_\alpha : \alpha \in \Delta_1\}) \cup (\cup\{A \cap V_{x_i} : 1 \le i \le n\})$ since $A \cap V_{x_i}$ is finite for each *i*, there exists a finite subset Δ_2 of Δ such that $(\cup\{A \cap V_{x_i} : 1 \le i \le n\}) \subset \{U_\alpha : \alpha \in \Delta\}$. Therefore, we obtain $A \subset \cup\{U_\alpha : \alpha \in \Delta_1 \cup \Delta_2\}$. This shows that A is λ-compact relative to X. Conversely let $\{V_\alpha : \alpha \in \Delta\}$ be any λ-open cover of X. We choose and fix one $\alpha_0 \in \Delta$ .Then $\cup\{V_\alpha : \alpha \in \Delta - \{\alpha_0\}\}$ is a λ-open cover of a $\mathcal{N}$-λ-closed set $X - V_{\alpha_0} \subset \cup\{V_\alpha : \alpha \in \Delta_0\}$.

Therefore, $X = \cup\{V_\alpha : \alpha \in \Delta_0 \cup \{\alpha_0\}\}$. This shows that X is λ-compact.

Theorem 3.7: Let (X, τ) be a topological space such that .Then is λ-compact if and only if is compact.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be any λ-open cover of $(X, \mathcal{N}\lambda O(X))$. For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $x \in V_{\alpha(x)}$. Since $V_{\alpha(x)}$ is $\mathcal{N}$-λ-open, there exist a λ-open set $U_{\alpha(x)}$ such that $x \in U_{\alpha(x)}$ and $U_{\alpha(x)} - V_{\alpha(x)}$ is finite. The family $\{U_{\alpha(x)} : x \in X\}$ is a λ-open cover of (X, τ). Since (X, τ) is λ-compact, there exists a finite subset $x_1, x_2, \ldots x_n$, such that $X = \cup\{U_{\alpha(x_i)} : i \in F\}$, where $F = \{1,2,\ldots n\}$. Now, we have $X = \bigcup_{i\in F}((U_{\alpha(x_i)} - V_{\alpha(x_i)}) \cup V_{\alpha(x_i)}) = \bigcup_{i\in F}(U_{\alpha(x_i)} - V_{\alpha(x_i)}) \cup \bigcup_{i\in F} V_{\alpha(x_i)}$. For each x_i, $U_{\alpha(x_i)} - V_{\alpha(x_i)}$ is a finite set and there exists a finite subset $\Delta(x_i)$ of Δ such that $(U_{\alpha(x_i)} - V_{\alpha(x_i)}) \cap X \subseteq \cup\{V_\alpha : \alpha \in \Delta(x_i)\}$. Therefore, we have $X = \left(\bigcup_{i\in F}(\cup\{V_\alpha : \alpha \in \Delta(x_i)\})\right) \cup \left(\bigcup_{i\in F} V_{\alpha(x_i)}\right)$. Hence $\mathcal{N}\lambda O(X)$ is compact. Conversely, let $\mathcal{U}$ be a λ-open cover of (X, τ). Then $\mathcal{U} \subseteq \mathcal{N}\lambda O(X)$. Since $(X, \mathcal{N}\lambda O(X))$ is compact, there exists a finite subcover of $\mathcal{U} \subseteq \mathcal{N}\lambda O(X)$. Since $(X, \mathcal{N}\lambda O(X))$ is compact, there exists a finite subcover of $\mathcal{U}$ for X. Hence (X, τ) is λ-compact.

Theorem 3.8: An $\mathcal{N}$-λ-closed subset of a λ-compact space X is λ-compact relative to X.

Proof: Let A be a $\mathcal{N}$-λ-closed subset of X. Let $\{U_\alpha : \alpha \in \Delta\}$ be a cover of A by λ-open sets of X. Now for each $x \in X - A$ there is a λ-open set V_x such that $V_x - A$ is finite. Since $\{U_\alpha : \alpha \in \Delta\} \cup \{V_x : x \in X - A\}$ is a λ-open cover of X and X is λ-compact, there exist a finite subcover $\{U_{\alpha_i} : i \in N\} \cup \{V_{x_i} : i \in N\}$. Since $\bigcup_{i\in N}(V_{x_i} \cap A)$ is finite, so for each $x_j \in \cup(V_{x_i} \cap A)$ there is $U_{\alpha(x_i)} \in \{U_\alpha : \alpha \in \Delta\}$ such that $x_j \in U_{\alpha(x_i)}$ and $j \in N$. Hence $\{U_{\alpha_i} : i \in N\} \cup \{U_{\alpha(x_i)} : i \in N\}$ is a finite subcover of $\{U_\alpha : \alpha \in \Delta\}$ and it covers A. Therefore, A is λ-compact relative to X.

Corollary 3.9: If a topological space X is λ-compact and A is λ-closed, then A is λ-compact relative to X.

4. PRESERVATION THEOREMS

Definition 4.1: A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\mathcal{N}$-λ-continuous (resp. λ-continuous [1]) if the inverse image of every open subset of Y is $\mathcal{N}$-λ-open in X.

It is clear that every λ-continuous function is $\mathcal{N}$-λ-continuous but not conversely.

Example 4.2: Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a, c\}, X\}$. Clearly the identity function $f : (X, \tau) \to (X, \sigma)$ is $\mathcal{N}$-λ-continuous but not λ-continuous.

Theorem 4.3: A function $f : (X, \tau) \to (Y, \sigma)$ is $\mathcal{N}$-λ-continuous if and only if for each point x in X and each open set V in Y with $f(x) \in V$, there is an $\mathcal{N}$-λ-open set U in X such that $x \in U$, and $f(U) \subseteq V$.

Proof: Let V be an open set in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and thus there exist $U_x \in \mathcal{N}\lambda O(X)$ such that $x \in U_x$ and $f(U_x) \subseteq V$. Then $x \in U_x \subseteq f^{-1}(V) = \cup\{U_x : x \in f^{-1}(V)\}$ Then by Lemma 2.7 $f^{-1}(V)$ is $\mathcal{N}$-λ-open. Conversely, let $x \in X$ and V be an open set of Y containing $f(x)$. Then $x \in f^{-1}(V) \in \mathcal{N}\lambda O(X)$ since f is $\mathcal{N}$-λ-continuous. Let $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subseteq V$.

Theorem 4.4: Let $f : (X, \tau) \to (Y, \sigma)$ be a $\mathcal{N}$-λ-continuous function. If X is λ-compact, then Y is compact.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be an open cover of Y. Then, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a $\mathcal{N}$-λ-cover of X. Since X is λ-compact, by Corollary 3.9 there exist a finite subset Δ_0 of Δ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$ hence $Y = \cup\{V_\alpha : \alpha \in \Delta_0\}$. Therefore Y is compact.

Definition 4.5: A function $f : X \to Y$ is said to be strongly λ-open if the image of each λ-open subset of X is λ-open in Y.

Proposition 4.6: If $f : X \to Y$ is strongly λ-open, then the image of an $\mathcal{N}$-λ-open set of X is $\mathcal{N}$-λ-open in Y.

Proof: Let $f : X \to Y$ be strongly λ-open and W an $\mathcal{N}$-λ-open subset of X. For any $y \in f(W)$, there exist $x \in W$ such that $f(x) = y$. Since W is $\mathcal{N}$-λ-open, there exists a λ-open set U such that $x \in U$ and $U - W = C$ is finite. Since *f* is strongly λ-open, $f(U)$ is λ-open in Y such that $y = f(x) \in f(U)$ and $f(U) - f(W) \subseteq f(U - W) = f(C)$ is finite. Therefore, $f(W)$ is $\mathcal{N}$-λ-open in Y.

Definition 4.7: [2] A function $f : X \to Y$ is said to be λ-irresolute if the inverse image of each λ-open subset of Y is λ-open in X.

Proposition 4.8: If $f : X \to Y$ is a λ-irresolute injection and A is $\mathcal{N}$-λ-open in Y, then $f^{-1}(A)$ is $\mathcal{N}$-λ-open in X.

Proof: Assume that A is an $\mathcal{N}$-λ-open subset of Y. Let $x \in f^{-1}(A)$. Then $f(x) \in A$ and there exists a $\mathcal{N}$-λ-open set V containing $f(x)$ such that $V - A$ is finite. Since *f* is λ-irresolute, $f^{-1}(V)$ is a λ-open set containing *x*. Thus $f^{-1}(V) - f^{-1}(A) = f^{-1}(V - A)$ and it is finite. It follows that $f^{-1}(A)$ is $\mathcal{N}$-λ-open in X.

Definition 4.9: A function $f : X \to Y$ is said to be $\mathcal{N}$-λ-closed if $f(A)$ is $\mathcal{N}$-λ-closed in Y for each λ-closed set A of X.

It is clear that every strongly λ-closed function is $\mathcal{N}$-λ-closed but not conversely. The function *f* in Example 4.2 is $\mathcal{N}$-λ-closed but not strongly $\mathcal{N}$-λ-closed.

Theorem 4.10: If $f : X \to Y$ is an $\mathcal{N}$-λ-closed surjection such that $f^{-1}(y)$ is λ-compact relative to X for each $y \in Y$ and Y is λ-compact, then X is λ-compact.

Proof: Let $\{U_\alpha : \alpha \in \Delta\}$ be any λ-open cover of X. For each $y \in Y$, $f^{-1}(y)$ is λ-compact relative to X and there exists a finite subset $\Delta(y)$ of Δ such that $f^{-1}(y) \subset \cup\{U_\alpha : \alpha \in \Delta(y)\}$. Now, we put $U(y) = \cup\{U_\alpha : \alpha \in \Delta(y)\}$ and $V(y) = Y - f(X - V(y))$. Then, since *f* is $\mathcal{N}$-λ-closed, $V(y)$ is an $\mathcal{N}$-λ-open set in Y containing *y* such that $f^{-1}(V(y)) \subset U(y)$. Since $\{V(y) : y \in Y\}$ is an $\mathcal{N}$-λ-open cover of Y, by Corollary 3.9 there exists a finite subset $\{y_k : 1 \le k \le n\} \subseteq Y$ such that $Y = \bigcup_{k=1}^{n} V(y_k)$. Therefore,

$X = f^{-1}(y) = \bigcup_{k=1}^{n} f^{-1}(V(y_k)) \subseteq \bigcup_{k=1}^{n} U(y_k) = \bigcup_{k=1}^{n} \{U_\alpha : \alpha \in \Delta(y_k)\}$. This shows that X is λ-Compact.

Definition 4.11: A function $f : X \to Y$ is said to be $\mathcal{N}$-λ-continuous if for each $x \in X$ and each λ-open set V of Y containing $f(x)$, there exist an $\mathcal{N}$-λ-open set U of X containing *x* such that $f(U) \subseteq V$.

Theorem 4.12: Let $f : X \to Y$ be a $\mathcal{N}$-λ-continuous surjection from X onto to Y. If X is λ-compact, then Y is λ-compact.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be a λ-open cover of Y. For each $x \in X$, there exist $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since *f* is $\mathcal{N}$-λ-continuous, there exists an $\mathcal{N}$-λ-open set of X containing *x* such that $f(U_{\alpha(x)}) \subseteq (V_{\alpha(x)})$. So $\{U_{\alpha(x)} : x \in X\}$ is an $\mathcal{N}$-λ-open cover of the λ-compact space X, by Corollary 3.9 there exists a finite subset

$\{x_k : 1 \leq k \leq n\} \subseteq X$ such that $X = \bigcup_{k=1}^{n} U_{\alpha(x_k)}$. Therefore, $Y = f(X) = f\left(\bigcup_{k=1}^{n} U_{\alpha(x_k)}\right) \subseteq \bigcup_{k=1}^{n} V_{\alpha(x_k)}$. This shows that Y is λ-Compact.

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