

On Isomteries of subspaces of the Dirichlet space among the multiplication operators

K. Hedayatian*

Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran

(Received on: 17-08-12; Revised & Accepted on: 31-08-12)

ABSTRACT

Suppose that M is a nonzero closed subspace of the Dirichlet space such that $i^k M \subseteq M$ for some $k > 0$, where i is the identity function. We show that analytic symbols that yield isometric multiplication operator on M are the unimodular constants.

2000 Mathematics subject Classification: 31C25, 47B38.

Keywords and Phrases: Multiplication operators, isometries, subspaces, Dirichlet space.

MAIN RESULTS

The classical Dirichlet space D is the only Hilbert space, up to an isomorphism, among the conformally invariant Banach spaces on which the Möbius group acts boundedly. These Banach spaces, also includes the Bloch B , the little Bloch space B_0 , and all analytic Besov spaces $B^p, 1 \leq p < \infty$. Note that $B^2 = D$.

Linear surjective isometries of Bergman and Hardy spaces are characterized in [4], and of the Bloch space and the little Bloch spaces are identified by Cima and Wogen in [3]. Furthermore, all linear surjective isometries of Besov spaces, except the case $p = 2$, are described by Hornor and Jamison [6]. The Dirichlet space D , being a Hilbert space, has many isometries that are not characterized. Martin and Vukotic [7], have given necessary and sufficient conditions for a composition operator to be an isometry on the Dirichlet space. In this note, we describe all multiplication operators on certain subspaces of the Dirichlet space D , especially on D , which are isometries.

Recall that if B is the Bergman space on the open unit disc U , then

$$\|f\|_B^2 = \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} = \int_U |f|^2 dA, \quad f \in B$$

where dA denotes the normalized area measure. The norm $\|\cdot\|_D$ is defined by

$$\|f\|_D^2 = \sum_{n=0}^{\infty} (n+1)|\hat{f}(n)|^2 = \|f\|_{H^2}^2 + \|f\|_B^2.$$

where $\|\cdot\|_{H^2}$ is the norm on the Hardy space H^2 .

The Dirichlet space D is the collection of analytic functions that map U onto a region of finite area with the above norm. The area of $f(U)$ equals to

$$\int_U |\hat{f}|^2 dA.$$

Note that $D \subseteq H^2 \subseteq B$. For a good reference of Dirichlet spaces see [1]. It is well known that the functional e_z of evaluation at z on the space D is a bounded linear functional [9].

Corresponding author: K. Hedayatian*

Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran

Recall that a linear isometry of a Hilbert space is a linear operator T such that $\|Tf\| = \|f\|$ for all f in the space. A good source on linear isometries of spaces of analytic functions is [5, Chapter 4]. We also remark that an operator T on a Hilbert space H is a 2-isometry if

$$\|T^2f\|^2 - 2\|Tf\|^2 + \|f\|^2 = 0,$$

for all $f \in H$. A paper of Richter and Sundberg [8, Theorem 4.2] says that the symbols that yield 2-isometric multiplication operators on the Dirichlet space must be inner functions. Any isometry is certainly a 2-isometry and so the isometric multiplication operators must be given by multiplication by inner functions. Since the multiplication by a non-constant inner function strictly increases the Dirichlet norm [2], the symbol must be a unimodular constant. In the following theorem we show that this holds also for many subspaces of the Dirichlet space.

Theorem. Let M be a nonzero closed subspace of the Dirichlet space such that $i^k M \subseteq M$ for some $k > 0$, where $i(z) = z$ for all z in the open unit disc U . Then the multiplication operator $M_\varphi : M \rightarrow M$ defined by $M_\varphi f = \varphi f$ is an isometry if and only if there is a constant α such that $\varphi(z) = e^{i\alpha}, \forall z \in D$.

Proof. Let M_φ be an isometry of M and f be nonzero element in M . If f has a zero of order n at $z \in U$ and φf has a zero of order m at z , then $(j-1)n \leq jm$ for every $j \geq 1$ because $f^{j-1}(\varphi f^j) = (\varphi f)^j$ and φf^j is analytic on U . Therefore, $\frac{j-1}{j} \leq \frac{m}{n}$ for $j = 1, 2, 3, \dots$ letting $j \rightarrow \infty$, we conclude that $n \leq m$, which

implies that $\varphi = \frac{\varphi f}{f}$ is analytic on U . Moreover, from the fact that $\|M_\varphi\| = 1$ we conclude

$$|\varphi^n(z)f(z)| = |e_z(\varphi^n f)| \leq \|e_z\| \|M_{\varphi^n f}\| \leq \|e_z\| \|f\|$$

for each $n \geq 1$ and all z in U , and so

$$|\varphi(z)| |f(z)|^{1/n} \leq (\|e_z\| \|f\|)^{1/n}.$$

Letting $n \rightarrow \infty$ we see that $|\varphi(z)| \leq 1$ for all z such that $f(z) \neq 0$, but such z 's are dense; so $|\varphi(z)| \leq 1$ on U .

If $g = \sum_{n=0}^{\infty} \hat{g}(n)z^n$ is in the Dirichlet space then

$$\begin{aligned} \|i^k g\|_D^2 - \|g\|_D^2 &= \sum_{n=k}^{\infty} (n+1) |\hat{g}(n-k)|^2 - \sum_{n=0}^{\infty} (n+1) |\hat{g}(n)|^2 \\ &= \sum_{n=0}^{\infty} (n+k+1) |\hat{g}(n)|^2 - \sum_{n=0}^{\infty} (n+1) |\hat{g}(n)|^2 \\ &= k \|g\|_{H^2}^2. \end{aligned}$$

Since

$$\|\varphi^n(i^k f)\|_D^2 - \|\varphi^n f\|_D^2 = \|i^k f\|_D^2 - \|f\|_D^2, \quad n \geq 1$$

applying the above formula for $g = \varphi^n f$ we get

$$\|\varphi^n f\|_{H^2}^2 = \|f\|_{H^2}^2, \quad n \geq 1. \tag{1}$$

Denote the unit circle by ∂U . the radial limit function of an arbitrary g in D , defined on ∂U , by g^* , and let $dm = \frac{d\theta}{2\pi}$ be the normalized arc length measure on ∂U . If $A = \{e^{i\theta} : |\varphi^*(e^{i\theta})| < 1\}$ and $B = \{e^{i\theta} : |\varphi^*(e^{i\theta})| = 1\}$ then since $|\varphi^*| \leq 1$ a.e. $[m]$, (1) implies that

$$\int_A |(\varphi^*)^n f^*|^2 dm = \int_A |f^*|^2 dm, \quad n \geq 1.$$

Applying the dominated convergence theorem, we conclude that $\int_A |f^*|^2 dm = 0$. Since f is not identically zero, then at almost all points of ∂U , $f^*(e^{i\theta}) \neq 0$. Thus $m(A) = 0$. This implies that $|\varphi^*(e^{i\theta})| = 1$ a.e. $[m]$, and consequently, φ is an inner function. If φ is not a constant, then it follows from Carleson's representation for Dirichlet integral, [2], that $\|(\varphi f)'\|_B > \|f'\|_B$. But $\|\varphi f\|_D = \|f\|_D$ and $\|\varphi f\|_{H^2} = \|f\|_{H^2}$, and so we get a contradiction. The converse is obvious.

Recall that an operator T on a Hilbert space H is called a unitary operator if T is a surjective linear isometry. As a direct consequence of our theorem we have the following.

Corollary. Let M be a nonzero closed subspace of the Dirichlet space such that $i^k M \subseteq M$ for some $k > 0$, where i is the identity function. Then the multiplication operator M_φ on M is an isometry if and only if it is a unitary operator.

We conclude with a question:

Is the above theorem true for every nonzero subspaces of the Dirichlet space?

ACKNOWLEDGMENT

This research was in part supported by a grant from Shiraz University Research Council.

REFERENCES

[1] L. Brown and A. L. Shields, Cyclic vectors in the Dirichlet space, *Trans. Amer. Math. Soc.* **285** (1984), 269-304
 [2] L. Carleson, A representation formula for the Dirichlet integral, *Math. Z.*, **73** (1960), 190-196.
 [3] J. A. Cima and W. R. Wogen, On isometries of the Bloch space, *Illinois J. Math.* **24** (1980), 313-316.
 [4] P. L. Duren and A. P. Schuster, Bergman Spaces, *Mathematical Surveys and Monographs*, **100**, American Mathematical Society, Providence, RI 2004.
 [5] R. J. Fleming and J. E. Jamison, Isometries on Banach spaces. function spaces, Chapman & Hall / CRC Monographs and Surveys in Pure and Applied Mathematics **129**, 2003.
 [6] W. Hornor and J. E. Jamison, Isometries of some Banach spaces of analytic functions, *Integral Equations Operator Theory* **41** (2001), no. 4, 410-425.
 [7] M. J. Marin and D. Vukotic, Isomeries of the Dirichlet space among the composition operators, *Proc. Amer. Math. Soc.* **134** (2006), 1701-1705.
 [8] S. Richter and C. Sundberg, A formula for local Dirichlet integral, *Michigan Math. J.* **38** (1991), 355-379.
 [9] G. D. Taylor, Multipliers on D_α , *Trans. Amer. Math. Soc.* **123** (1966), 229-240.

Source of support: Research partially supported by a grant from Shiraz University Research Council, Conflict of interest: None Declared