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On Isomteries of subspaces of the Dirichlet space among the multiplication operators

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ABSTRACT

Suppose that M is a nonzero closed subspace of the Dirichlet space such that $i^k M \subseteq M$ for some $k>0$, where i is the identity function. We show that analytic symbols that yield isometric multiplication operator on M are the unimodular constants.

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MAIN RESULTS

The classical Dirichlet space D is the only Hilbert space, up to an isomorphism, among the conformally invariant Banach spaces on which the M$\ddot{o}$bius group acts boundedly. These Banach spaces, also includes the Bloch B, the little Bloch space B_0, and all analytic Besov spaces $B^p, 1 \le p < \infty$. Note that $B^2 = D$.

Linear surjective isometries of Bergman and Hardy spaces are characterized in [4], and of the Bloch space and the little Bloch spaces are identified by Cima and Wogen in [3]. Furthermore, all linear surjective isometries of Besov spaces, except the case $p = 2$, are described by Hornor and Jamison [6]. The Dirichlet space D, being a Hilbert space, has many isometries that are not characterized. Martin and Vukotic [7], have given necessary and sufficient conditions for a composition operator to be an isometry on the Dirichlet space. In this note, we describe all multiplication operators on certain subspaces of the Dirichlet space D, especially on D, which are isometries.

Recall that if B is the Bergman space on the open unit disc U, then

$$||f||_B^2 = \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} = \int_U |f|^2 dA, \qquad f \in B$$

where dA denotes the normalized area measure. The norm $||.||_D$ is defined by

$$||f||_D^2 = \sum_{n=0}^{\infty} (n+1)|\hat{f}(n)|^2 = ||f||_{H^2}^2 + ||f'||_B^2.$$

where $||.||_{H^2}$ is the norm on the Hardy space H^2.

The Dirichlet space D is the collection of analytic functions that map U onto a region of finite area with the above norm. The area of $f(U)$ equals to

$$\int_U |f'|^2 \, dA.$$

Note that $D \subseteq H^2 \subseteq B$. For a good reference of Dirichlet spaces see [1]. It is well known that the functional e_z of evaluation at z on the space D is a bounded linear functional [9].

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Recall that a linear isometry of a Hilbert space is a linear operator T such that $\|Tf\|=\|f\|$ for all f in the space. A good source on linear isometries of spaces of analytic functions is [5, Chapter 4]. We also remark that an operator T on a Hilbert space H is a 2-isometry if

$$\|T^2 f\|^2 - 2\|Tf\|^2 + \|f\| = 0,$$

for all $f \in H$. A paper of Richter and Sundberg [8, Theorem 4.2] says that the symbols that yield 2-isometric multiplication operators on the Dirichlet space must be inner functions. Any isometry is certainly a 2-isometry and so the isometric multiplication operators must be given by multiplication by inner functions. Since the multiplication by a non-constant inner function strictly increases the Dirichlet norm [2], the symbol must be a unimodular constant. In the following theorem we show that this holds also for many subspaces of the Dirichlet space.

Theorem. Let M be a nonzero closed subspace of the Dirichlet space such that $i^k M \subseteq M$ for some $k > 0$, where $i(z) = z$ for all z in the open unit disc *U*. Then the multiplication operator $M_\varphi : M \to M$ defined by $M_\varphi f = \varphi f$ is an isometry if and only if there is a constant α such that $\varphi(z) = e^{i\alpha}, \forall z \in D$.

Proof. Let M_φ be an isometry of M and f be nonzero element in M. If f has a zero of order n at $z \in U$ and φf has a zero of order m at z, then $(j-1)n \le jm$ for every $j \ge 1$ because $f^{j-1}(f\varphi^j) = (f\varphi)^j$ and $f\varphi^j$ is analytic on *U*. Therefore, $\frac{j-1}{j} \le \frac{m}{n}$ for $j = 1, 2, 3, \cdots$ letting $j \to \infty$, we conclude that $n \le m$, which implies that $\varphi = \frac{\varphi f}{f}$ is analytic on *U*. Moreover, from the fact that $\|M_\varphi\| = 1$ we conclude

$$|\varphi^n(z) f(z)| = |e_z(\varphi^n f)| \le \|e_z\| \|M_{\varphi^n} f\| \le \|e_z\| \|f\|$$

for each $n \ge 1$ and all z in *U*, and so

$$|\varphi(z)| |f(z)|^{1/n} \le (\|e_z\| \|f\|)^{1/n}.$$

Letting $n \to \infty$ we see that $|\varphi(z)| \le 1$ for all z such that $f(z) \ne 0$, but such $z's$ are dense; so $|\varphi(z)| \le 1$ on *U*.

If $g = \sum_{n==0}^{\infty} \hat{g}(n) z^n$ is in the Dirichlet space then

$$\begin{aligned}\|i^k g\|_D^2 - \|g\|_D^2 &= \sum_{n=k}^{\infty} (n+1)|\hat{g}(n-k)|^2 - \sum_{n=0}^{\infty} (n+1)|\hat{g}(n)|^2 \\ &= \sum_{n=0}^{\infty} (n+k+1)|\hat{g}(n)|^2 - \sum_{n=0}^{\infty} (n+1)|\hat{g}(n)|^2 \\ &= k\|g\|_{H^2}^2 .\end{aligned}$$

Since

$$\|\varphi^n (i^k f)\|_D^2 - \|\varphi^n f\|_D^2 = \|i^k f\|_D^2 - \|f\|_D^2, \qquad n \ge 1$$

applying the above formula for $g = \varphi^n f$ we get

$$\|\varphi^n f\|_{H^2}^2 = \|f\|_{H^2}^2, \qquad n \ge 1. \tag{1}$$

Denote the unit circle by ∂U. the radial limt function of an arbitrary g in D, defined on ∂U, by g^*, and let $dm = \frac{d\theta}{2\pi}$ be the normalized arc length measure on ∂U, If $A = \{e^{i\theta} : |\varphi^*(e^{i\theta})| < 1\}$ and $B = \{e^{i\theta} : |\varphi^*(e^{i\theta})| = 1\}$ then since $|\varphi^*| \le 1$ a.e. $[m]$, (1) implies that

$$\int_A |(\varphi^*)^n f^*|^2 \, dm = \int_A |f^*|^2 \, dm, \qquad n \ge 1.$$

Applying the dominated convergence theorem, we conclude that $\int_A |f^*|^2 \, dm = 0$. Since f is not identically zero, then at almost all points of ∂U, $f^*\left(e^{i\theta}\right) \neq 0$ Thus $m(A) = 0$. This implies that $|\varphi^*(e^{i\theta})| = 1$ a.e. $[m]$, and consequently, φ is an inner function. If φ is not a constant, then it follows from Carleson's representation for Dirichlet integral, [2], that $\|(\varphi f)'\|_B > \|f'\|_B$. But $\|\varphi f\|_D = \|f\|_D$ and $\|\varphi f\|_{H^2} = \|f\|_{H^2}$, and so we get a contradiction. The converse is obvious.

Recall that an operator T on a Hilbert space H is called a unitary operator if T is a surjective linear isometry. As a direct consequence of our theorem we have the following.

Corollary. Let M be a nonzero closed subspace of the Dirichlet space such that $i^k M \subseteq M$ for some $k > 0$, where i is the identity function. Then the multiplication operator M_φ on M is an isometry if and only if it is a unitary operator.

We conclude with a question:

Is the above theorem true for every nonzero subspaces of the Dirichlet space?

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