International Journal of Mathematical Archive-3(8), 2012, 3113-3117
Available online through www.ijma.info ISSN 2229 – 5046

FRACTIONAL OUT-DOMINATION NUMBER OF AN OUT-TREE

K. Muthu Pandian*
Associate Professor of Mathematics, Government Arts College, Melur 625 106, India

(Received on: 18-08-12; Accepted on: 31-08-12)

ABSTRACT

An out-dominating function (ODF) of a digraph $D=(V,A)$ is a function $f:V\to[0,1]$ such that $\sum\limits_{u\in N^+[v]} f(u)\geq 1$ for all $v\in V$, where $N^+[v]$ consists of v with all vertices adjacent from it. For a real valued function $f:V\to R$, the weight of f is $|f|=\sum\limits_{v\in V} f(v)$. The fractional out-domination number of a digraph D, denoted $\gamma_{fo}(D)$, equals the minimum weight of an ODF of D. In this paper, we develop a method for finding $\gamma_{fo}(D)$ for an out-tree that satisfies specific condition. A supporting proof for this method is also given. We also apply this method to the out-trees directed path and directed star.

Keywords: *Digraphs, Out-dominating function, Minimal out-dominating function, fractional out-domination number $\gamma_{fo}(D)$, out-tree, directed path, directed star.*

Subject Classification: *MSC. 05C20.*

1. INTRODUCTION

We use Harary[3] for notation and terminology which are not defined here. The concept of dominating function and fractional domination number have been introduced in [6]. A dominating function (DF) of a Graph $G=(V,E)$ is a function $f:V\to[0,1]$ such that $\sum\limits_{u\in N[v]} f(u)\geq 1$ for all $v\in V$,

where $N[v]=\{u\in V/u \text{ is adjacent with } v\}\cup\{v\}$. A DF f is called minimal dominating function (MDF) if there is no DF g of G such that $g(v)\leq f(v)$ for all $v\in V$ and $g(v_0)\neq f(v_0)$ for some $v_0\in V$. For a real-valued function $f:V\to R$, the weight of f is $|f|=\sum\limits_{v\in V} f(v)$ and $S\subseteq V$, $f(S)=\sum\limits_{v\in S} f(v)$ so $|f|=f(V)$. The boundary set B_f and the positive set P_f of a DF f are defined by $B_f=\{v\in V/f(N[v])=1\}$ and $P_f=\{v\in V/f(v)>0\}$.

Let A and B be the subsets of V. We say that A dominates B and it can be written as $A\to B$, if every vertex in $B\setminus A$ is adjacent to some vertex in A. In this connection, the following theorem(Theorem 1.1) provides a necessary and sufficient condition for a DF to be a MDF.

Theorem 1.1. [2] A DF f of G is a MDF if and only if $B_f\to P_f$.

For a graph G, the fractional domination number $\gamma_f(G)$ is defined by $\gamma_f(G)$=min{$|f|/f$ is a MDF of G}. Although domination and other related concepts have been extensively studied for undirected graphs, the respective analogue on digraphs have not received much attention. Of course, a survey of results on domination in digraphs by Ghoshal, Lasker, and Pillone is found in Chapter 15 of Haynes et al. [4], but most of the results in this survey chapter deals with the concepts of kernels and solutions in digraphs and also on dominations in tournaments. For a survey of dominating functions, we also refer the research reviews by Haynes et al. [5]. As an initiation of the present research work, we already transferred the concept of dominating function (DF) and fractional domination number $\gamma_f(G)$ to digraphs, called out-dominating function and fractional out-domination number $\gamma_{fo}(D)$ in [7]. In continuation, a method has been developed for finding $\gamma_{fo}(D)$ for an out-tree that satisfies specific condition and it has been presented in the present paper along with its specific application.

Corresponding author: *K. Muthu Pandian**
Associate Professor of Mathematics, Government Arts College, Melur 625 106, India

2. OUT-DOMINATING FUNCTION

In this paper, we deal with digraphs which admit no self loops and no multiple arcs. Let D be a digraph with vertex set V arc set A. Either v is adjacent from u or u is adjacent to v if (u, v) is an arc of D. The out-degree od(v) of a vertex v is the number of vertices that are adjacent from it and the in-degree id(v) is the number of vertices adjacent to it. Let $N^+(v)$ denote the set of all vertices of D which are adjacent from v. Let $N^+[v] = N^+(v) \cup \{v\}$.

Definition 2.1. [7] An out-dominating function (ODF) of a digraph $D = (V, A)$ is a function $f : V \to [0, 1]$ such that $\sum_{u \in N^+[v]} f(u) \geq 1$ for all $v \in V$.

Definition 2.2. [7] An ODF f is called a minimal ODF if there is no ODF g of D such that $g(v) \leq f(v)$ for all $v \in V$ and $g(v_0) \neq f(v_0)$ for some $v_0 \in V$.

Definition 2.3. [7] The fractional out-domination number, denoted $\gamma_{fo}(D)$, is defined by $\gamma_{fo}(D)$=min{$|f|/f$ is a minimal out-dominating function of D}.

Definition 2.4. [7] Let A and B be two subsets of V. We say that A out-dominates B and write $A \xrightarrow{out} B$ if every vertex $u \in B \setminus A$ is adjacent from some vertex in A.

Notation 2.5.
1. $B_f^+ = \{v \in V / f(N^+[v]) = 1\}$

2. $P_f = \{v \in v / f(v) > 0\}$

Theorem 2.6. [7] An out-dominating function f of a digraph D is a minimal out-dominating function if and only if $B_f^+ \xrightarrow{out} P_f$.

Definition 2.7. [3] A source in D is a vertex by which all other vertices can be reached.

Definition 2.8. [3] A semi cycle is an alternating sequence $v_0, x_1, v_1, \ldots, v_n = v_0$ of distinct vertices and arcs starting and ending at v_0, but each arc x_i may be either (v_{i-1}, v_i) or (v_i, v_{i-1}).

So, in an out-tree, the point with in-degree 0 is called source or arborescence.

Theorem 2.10 [3] A weak digraph is an out-tree if and only if exactly one point has in-degree 0 and all other points in-degree 1.

3. A METHOD FOR FINDING $\gamma_{fo}(D)$ OF AN OUT-TREE

In this method, the vertices of the out-tree are grouped into m different levels as follows:

1. Identify the vertices with out-degree 0. Let them be $v_{11}, v_{12}, \ldots, v_{1n_1}$. They can be called as level 1 vertices.

2. Identify the vertices which are adjacent to at least any one or more of the level 1 vertices. Let them be $v_{21}, v_{22}, \ldots, v_{2n_2}$ and they can be called as level 2 vertices.

3. Identify the vertices which are adjacent to at least any one or more of the level 2 vertices and not equal to any of the level 2 vertices. Let them be $v_{31}, v_{32}, \ldots, v_{3n_3}$ and they can be called as level 3 vertices.

4. Identify the vertices which are adjacent to at least any one or more of the level 3 vertices and not equal to any of the level 3 vertices.. Let them be $v_{41}, v_{42}, \ldots, v_{4n_4}$ and they can be called as level 4 vertices.

These steps are to be proceeded, until all the vertices are identified. It is assumed that the vertices in an odd level are not adjacent to vertices in the same as well as any other odd level. As a consequence, $\gamma_{fo}(D)$ shall be equal to number of vertices in odd levels.

(i.e.),$\gamma_{fo}(D) = \begin{cases} n_1 + n_3 + \ldots + n_m & \text{m is odd} \\ n_1 + n_3 + \ldots + n_{m-1} & \text{m is even} \end{cases}$

A supporting proof for this result is presented in the following theorem.

Theorem 3.1. Let D be an out-tree. The vertices of D are grouped into m levels as per our discussion in this section. It is assumed that the vertices in an odd level are not adjacent to the vertices in the same as well as any other odd level. Then $\gamma_{fo}(D)$ shall be equal to the number of vertices in odd levels

(i.e.), $\gamma_{fo}(D) = \begin{cases} n_1 + n_3 + \ldots + n_m & \text{m is odd} \\ n_1 + n_3 + \ldots + n_{m-1} & \text{m is even} \end{cases}$, where n_i is the number of vertices in the i[th] level.

Proof: Let $v_{i1}, v_{i2}, \ldots, v_{in_i}$ be the vertices in the i[th] level, $i = 1, 2, \ldots, m$.

Let $f(v_{i1}) = 1, f(v_{i2}) = 1, \ldots, f(v_{in_i}) = 1$ if i is odd and let $f(v_{i1}) = 0, f(v_{i2}) = 0, \ldots, f(v_{in_i}) = 0$ if i is even.

Since $od(v_{11}) = 0, od(v_{12}) = 0, \ldots, od(v_{1n_1}) = 0$, we get
$N^+[v_{11}] = v_{11}, N^+[v_{12}] = v_{12}, \ldots, N^+[v_{1n_1}] = v_{1n_1}$.

Since $f(v_{11}) = 1, f(v_{12}) = 1, \ldots, f(v_{1n_1}) = 1$, it follows that
$f(N^+[v_{11}]) = 1, f(N^+[v_{12}]) = 1, \ldots, f(N^+[v_{1n_1}]) = 1$.

Every vertex in level 2 is adjacent to at least one vertex in level 1. Therefore,

$f(N^+[v_{21}]) \geq 1, f(N^+[v_{22}]) \geq 1, \ldots, f(N^+[v_{2n_2}]) \geq 1$.

Since $f(v_{31}) = 1, f(v_{32}) = 1, \ldots, f(v_{3n_3}) = 1$, we get
$f(N^+[v_{31}]) \geq 1, f(N^+[v_{32}]) \geq 1, \ldots, f(N^+[v_{3n_3}]) \geq 1$.

Continuing the process for all the levels, we get $f(N^+[v]) \geq 1$ for all $v \in V$. Since f values at odd level vertices are 1 and f values at even level vertices are 0, P_f= all vertices at odd levels. If v is a vertex in an odd level, $N^+[v]$ shall contain no vertex in odd levels other than v. Therefore, for a vertex at odd level, $f(N^+[v]) = 1$. Hence, $P_f \subseteq B_f^+$. (i.e.) $P_f \setminus B_f^+ = \phi$. (i.e.) $B_f^+ \xrightarrow{out} P_f$. So, by theorem 2.6, f is a minimal out-dominating function of D. Therefore, $\gamma_{fo}(D) \leq |f|$ and $|f|$= Number of vertices at odd levels.

So, $\gamma_{fo}(D) \leq \begin{cases} n_1 + n_3 + \ldots + n_m & \text{m is odd} \\ n_1 + n_3 + \ldots + n_{m-1} & \text{m is even} \end{cases}$ (1)

Let f be an out-dominating function of D. Since $od(v_{11}) = 0, od(v_{12}) = 0, \ldots, od(v_{1n_1}) = 0$, we get $f(v_{11}) = 1, f(v_{12}) = 1, \ldots, f(v_{1n_1}) = 1$.

Now,
$|f| = (f(v_{11}) + f(v_{12}) + \ldots + f(v_{1n_1})) + (f(v_{21}) + f(v_{22}) + \ldots + f(v_{2n_2})) + \ldots + (f(v_{m1}) + f(v_{m2}) + \ldots + f(v_{mn_m}))$ (2)

The vertices in an odd level are not adjacent to vertices in the same level. In addition, they are not adjacent to vertices in any other odd level. If v is a vertex in an odd level, $N^+[v]$ shall contain no vertex in odd levels other than v. Also, $N^+[u] \cap N^+[v] = \phi$ for any two vertices u and v. For , if $N^+[u] \cap N^+[v] \neq \phi$, let $w \in N^+[u] \cap N^+[v]$.

Then , w is adjacent from both u and v. Therefore, $id(w) \geq 2$. This is a contradiction as D is an out-tree.

Case (i): m is odd

From (1),

$|f| = 1+1+\ldots+n_1\text{times}+(f(N^+[v_{31}])+f(N^+[v_{32}])+\ldots+f(N^+[v_{3n_3}])+(f(N^+[v_{51}])+ f(N^+[v_{52}])+\ldots+f(N^+[v_{5n_5}])+\ldots+(f(N^+[v_{m1}])+f(N^+[v_{m2}])+\ldots+f(N^+[v_{mn_m}])+f$

values at some vertices at even levels. (Reasoning: For a vertex v in an odd level, $N^+[v]$ shall contain no vertex in odd levels other than v and $N^+[u]\cap N^+[v]=\phi$ for any two vertices u and v).

Therefore,

$|f| \geq (1+1+\ldots+n_1\text{times})+(1+1+\ldots+n_3\text{times})+\ldots+(1+1+\ldots+n_m\text{times})$(i.e.) $|f| \geq n_1+n_3+\ldots+n_m$ for any out-dominating function f of D.

Therefore, $\gamma_{fo}(D) \geq n_1+n_3+\ldots+n_m$

Case (ii): m is even

This is similar to case (i) and it can be proved that $\gamma_{fo}(D) \geq n_1+n_3+\ldots+n_{m-1}$.

Hence $\gamma_{fo}(D) \geq \begin{cases} n_1+n_3+\ldots+n_m & \text{m is odd} \\ n_1+n_3+\ldots+n_{m-1} & \text{m is even}\end{cases}$ (3)

Therefore, from (1) and (3) $\gamma_{fo}(D) = \begin{cases} n_1+n_3+\ldots+n_m & \text{m is odd} \\ n_1+n_3+\ldots+n_{m-1} & \text{m is even}\end{cases}$

Note 3.2 The method developed in this section in terms of number of levels and oddness or evenness of the number of levels is applied for some special out-trees.

Corollary 3.3 For a directed path $\overrightarrow{P_m}$, $\gamma_{fo}(\overrightarrow{P_m}) = \left\lceil \frac{m}{2} \right\rceil$, where $\lceil x \rceil$ is the smallest integer such that $x \leq \lceil x \rceil$

Proof: Let $\overrightarrow{P_m}=(V,A)$, where $V=\{v_1,v_2,\ldots,v_m\}$ and $A=\{(v_1,v_2),(v_2,v_3),\ldots,(v_{m-1},v_m)\}$ and this is clearly an out-tree.

1. Identify the vertices with out-degree 0. v_m is the only vertex with out-degree 0. This is the level 1 vertex and therefore $n_1=1$
2. Identify the vertex which is adjacent to v_m. v_{m-1} is the vertex adjacent to v_m. This is the level 2 vertex and therefore $n_2=1$.
3. Identify the vertex which is adjacent to v_{m-1}. v_{m-2} is the vertex adjacent to v_{m-1}. This is the level 3 vertex and therefore $n_3=1$

Proceeding the process as above, the last level (i.e the m^{th} level) vertex is v_1. Therefore, $n_m=1$.

Clearly the vertices in an odd level are not adjacent to vertices in the same as well as any other odd level.

Case (i): m is odd

$$\gamma_{fo}(\overrightarrow{P_m}) = n_1+n_3+\ldots+n_m = \frac{m+1}{2}$$

Case (ii): m is even

$$\gamma_{fo}(\overrightarrow{P_m}) = n_1+n_3+\ldots+n_{m-1} = \frac{m}{2}$$

Hence, $\gamma_{fo}(\overrightarrow{P_m}) = \left\lceil \frac{m}{2} \right\rceil$

Corollary 3.4. For a directed star $\overrightarrow{K_{1,n}}$, $\gamma_{fo}(\overrightarrow{K_{1,n}})$

Proof: Let $\overrightarrow{K_{1,n}} = (V, A)$, where $V = \{v_0, v_1, v_2, \ldots, v_n\}$, and $A = \{(v_0, v_1), (v_0, v_2), \ldots, (v_0, v_n)\}$ and this is clearly an out-tree.

1. Identify the vertices with out-degree 0. They are $v_1, v_2, \ldots, v_n$. These are level 1 vertices and therefore $n_1 = n$.
2. Identify the vertices which are adjacent to at least any one or more of the level 1 vertices. v_0 is the only vertex adjacent to level 1 vertices. Therefore, v_0 is the level 2 vertex and also it is the last level. So, $n_2 = 1$.

Clearly, the vertices in an odd level are not adjacent to vertices in the same level and not adjacent to vertices in any other odd level.

Therefore, $\gamma_{fo}(\overrightarrow{K_{1,n}}) = n_1 + n_3 + \ldots + n_{m-1}$=$n_1$ (since m=2)
(i.e.)$\gamma_{fo}(\overrightarrow{K_{1,n}}) = n$.

Hence, the developed method could be applied effectively to the directed path and directed star.

4. CONCLUSION

In this paper, we have developed a method for finding $\gamma_{fo}(D)$ for an out-tree that satisfies specific condition. We tried to develop a method for finding $\gamma_{fo}(D)$ for any out-tree, but so far without success. So, we conclude the paper with the following open problem.

Problem 4.1 Develop a method for finding $\gamma_{fo}(D)$ for any out-tree D

REFERENCES

[1] Chartand, G., Lesniak. L.: Graphs and Digraphs, Chapman and Hall, CRC, 4th edition 2005.

[2] Cockayne, E.J., Frickle, G., Hedetniemi, S,T., Mynhardt, C,M.: Properties of minimal dominating functions of graph, Ars Combin 41(1995), 107-115.

[3] Harary, F.: Graph Theory, Addison Wesley Publishing Company, inc, USA, 1969.

[4] Haynes, T, W., Hedetniemi, S.T., Slater, P, J: Domination in graphs – Advanced topics, Marcel Dekker, Inc, New York, 1998.

[5] Haynes, T,W., Hedetniemi, S,T., Slater, P,J: Fundamentals of domination in graphs, Marcel Dekker, Inc, New York, 1998.

[6] Hedetiniemi,S,M., Hedetniemi,S,T., Wimer, T,V.: Linear time resource allocation algorithm for trees. Technical report URJ-014, Department of Mathematics, Clemson University, 1987,

[7] Muthu Pandian,K., Kamarj,M: Directed vertex dominating function ,International Journal of Combinatorial Graph Theory and application, Vol.4, No.2, (July-December 2011), pp. 89-101 Global Research Publications New Delhi (India)New Delhi.

Source of support: Nil, Conflict of interest: None Declared