International Journal of Mathematical Archive-3(8), 2012, 3105-3107
Available online through www.ijma.info ISSN 2229 – 5046

ON NATIVIDAD'S FORMULA FOR SOLVING THE MISSING TERMS OF A RECURRENCE SEQUENCE

Julius Fergy T. Rabago†*

†Faculty, Department of Mathematics and Physics, College of Arts and Sciences, Central Luzon State University Science City of Muñoz 3120, Nueva Ecija, Philippines

(Received on: 02-06-12; Accepted on: 20-06-12)

ABSTRACT

This paper presents a formal proof of the results found on [1] and [2] of Natividad.

Keywords and Phrases: *Binet's formula, Fibonacci-like Sequence, Pell sequence, Pell-like sequence, Pell means, missing terms.*

2010 Mathematics Subject Classifications: *11B39, 11B50, 11B99.*

1. INTRODUCTION

Natividad [1] presents a formula for solving the missing terms of a Fibonacci-like sequence given a first term a and a last term b. Also, he provides a solution for solving Pell-means, *see* [2]. He shows that the formula for the first term, denoted by x_1, for solving the missing terms of a Fibonacci-like sequence with first term a and last term b is given by,

$$x_1 = \frac{b - F_n a}{F_{n+1}} ,$$

where n is the number of missing terms and F_n is the n-th term of the Fibonacci sequence.

On the other hand, the formula for solving Pell means can be found using the formula for x_1 defined by

$$x_1 = \frac{b - P_n a}{P_{n+1}} ,$$

where P_n is the n-th term of the Pell sequence, as shown also by Natividad.

Natividad have derived the formula for the second term, $W_1 = x_1$, of the sequence dependent on the first and last term by looking at each case of the number of missing terms. He also provided a table for the coefficient of a in the numerator and the coefficient of the denominator and looked for a pattern as the number of missing terms increases.

In this note, we will derive the formula for solving the missing terms of the sequence by using the Binet's formula of the recurrence relation and using the fact that b is the $(n+1)^{\text{th}}$ term of the sequence with a as the first term. Thus, here we present the same results shown by Natividad but of different approach.

2. PRELIMINARIES

From [1] and [2], we have the following definition:

Definition 2.1. If $a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n, b$ is a Fibonacci-like (or a Pell-like) sequence, then $x_1, x_2, x_3, \ldots, x_{n-1}$, and x_n are called Fibonacci means (or Pell means) between a and b.

Before we proceed for the formula on solving the missing terms of a Fibonacci-like sequence as well as the formula for solving Pell-means, we consider Horadam's [3] second-order linear recurrence sequence $\{W_n\}$ defined by the recurrence relation $W_{n+1} = pW_n + qW_{n-1}$, with $W_0 = a$, $W_1 = c$, where a, c and p, q are arbitrary real numbers for $n > 0$.

Corresponding author: *Julius Fergy T. Rabago†**
†Faculty, Department of Mathematics and Physics, College of Arts and Sciences, Central Luzon State University Science City of Muñoz 3120, Nueva Ecija, Philippines

The Binet's formula for the recurrence sequence $\{W_n\}$ is given by

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1}$$

where $A = c - a\beta$, $B = c - a\alpha$.

Take note that the generating function of $\{W_n\}$ is a quadratic equation of the form, $x^2=px+q$. Solving for its roots we obtain,

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2} \text{ and } \beta = \frac{p-\sqrt{p^2+4q}}{2}.$$

Notice that $\alpha + \beta = p$, $\alpha - \beta = \sqrt{p^2+4q}$ and $\alpha\beta = -q$.

The sole purpose of considering the second-order linear recurrence sequence $\{W_n\}$ is to use its two special cases. When $p = 1$ and $q = 1$, W_n is the n-th term of the Fibonacci-like sequence and when $p = 2$ and $q = 1$, W_n becomes the n-th term of the Pell-like sequence. In terms of recurrence relation, we have the following special cases:

(i) if $p = 1$ and $q = 1$, we will have a Fibonacci like-sequence defined by

$$W_{n+1} = F_{n+1}=F_n + F_{n-1} \quad \text{where} \quad W_0 = a,\ W_1 = c \tag{2}$$

(ii) if $p = 2$ and $q = 1$, we will obtain a Pell-like sequence given by

$$W_{n+1} = P_{n+1}=2P_n + P_{n-1} \quad \text{where} \quad W_0 = a,\ W_1 = c \tag{3}$$

In the following section we shall provide an explicit formula for solving the general term W_n of the recurrence relation (2) and (3) using equation (1) as part of the proof of the following formula: $x_1 = \frac{b-F_n}{F_{n+1}}$ for Fibonacci-like sequences and $x_1 = \frac{b-P_n}{P_{n+1}}$ for Pell-like sequences.

3. MAIN RESULTS

We will use the Binet's formula for the second-order linear recurrence sequence $\{W_n\}$ of the two special cases mentioned above to prove the following theorems.

Theorem 3.1. *For any real numbers a and b*

$$x_1 = \frac{b-F_n a}{F_{n+1}}, \tag{4}$$

where n is the number of missing terms, F_n is the n-th Fibonacci number and, a and b is defined as the first term and the last term of the sequence, respectively.

Proof. Let $W_0 = a$ be the first term of the sequence and b be the last term of the sequence. If n is the number of missing terms between a and b then, $b = W_{n+1}$. Now suppose that x_1 is the second term in the sequence, hence the Binet's formula for the sequence is given by

$$W_n = \frac{(x_1 - a\beta)\alpha^n - (x_1 - a\alpha)\beta^n}{\sqrt{5}}$$

where α is equal to φ known as the golden ratio and $\beta = 1 - \varphi$.

This would imply that,

$$b = \frac{(x_1 - a\beta)\alpha^{n+1} - (x_1 - a\alpha)\beta^{n+1}}{\sqrt{5}}$$

$$= \frac{(\alpha^{n+1} - \beta^{n+1})x_1 - (\alpha^{n+1}\beta - \alpha\beta^{n+1})a}{\sqrt{5}}$$

$$= \frac{(\alpha^{n+1} - \beta^{n+1})x_1 - (\alpha^n - \beta^n)a\alpha\beta}{\sqrt{5}}$$

$$= F_{n+1}x_1 + F_n a$$

This proves equation (4).

Theorem 3.2. *For any real numbers a and b*

$$x_1 = \frac{b - P_n a}{P_{n+1}}, \tag{5}$$

where n is the number of missing terms, P_n is the n-th term of the Pell sequence, and a and b is defined as the first term and the last term of the sequence, respectively.

Proof. The proof is similar to the previous theorem. Using the Binet's formula for the Pell-like sequence with first term $W_0 = a$ and last term b, we have

$$W_n = \frac{(x_1 - a\beta)\alpha^n - (x_1 - a\alpha)\beta^n}{2\sqrt{2}}$$

where α is equal to $1+\sqrt{2}$ known as the silver ratio and $\beta = 1 - \alpha$.

Letting n be the number of missing terms between a and b, we obtain

$$b = W_{n+1} = \frac{(x_1 - a\beta)\alpha^{n+1} - (x_1 - a\alpha)\beta^{n+1}}{2\sqrt{2}}$$

$$= \frac{(\alpha^{n+1} - \beta^{n+1})x_1 - (\alpha^{n+1}\beta - \alpha\beta^{n+1})a}{2\sqrt{2}}$$

$$= \frac{(\alpha^{n+1} - \beta^{n+1})x_1 - (\alpha^n - \beta^n)a\alpha\beta}{2\sqrt{2}}$$

$$= P_{n+1}x_1 + P_n a$$

Equation (5) follows.

ACKNOWLEDGEMENT

I would like to thank my brother Mr. Angel Joy T. Rabago for reviewing my research article.

REFERENCES

[1] Natividad, L.R., Deriving a formula in solving Fibonacci-like sequence, *International Journal of Mathematics and Scientific Computing* **1** (2011), #1 19-21.

[2] Natividad, L.R., On Solving Pell Means, *International Journal of Mathematical Archive* **12** (2011), #2 2736-2739.

[3] A. F. Horadam, Basic properties of certain generalized sequence of numbers, *Fibonacci Quart.* **3** (1965), 161-176.

Source of support: Nil, Conflict of interest: None Declared