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AN ITERATIVE REDUCED-BIAS ALGORITHM FOR A DUAL-FUSION VARIANT OF BERNSTEIN'S OPERATOR

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ABSTRACT

The celebrated Weierstrass Approximation Theorem (1885) heralded intermittent interest in polynomial approximation, which continues unabated even as of today. The great Russian mathematician Bernstein, in 1912, not only provided an interesting proof of the Weierstrass' theorem, but also displayed a sequence of the polynomials which approximate the given function f (x) ε C[0, 1]. This paper is motivated by the "Iterative Statistical Bias-Reduction Strategy" proposed in Sahai (2004) for the Bernstein's Polynomial Approximation operator. A 'Dual-Fusion' version of the Bernstein's Polynomial Operator is proposed. This version has a 'Systematic-Bias' in approximation which is much more accessible to the 'Iterative Algorithm' of 'Statistical Bias-Reduction' proposed in Sahai (2004).The potential of the aforesaid improvement algorithm is tried to be brought forth and illustrated through an empirical study for which the function is assumed to be known in the sense of simulation.

Keywords: *Approximation; Bernstein operator; Simulated empirical study*

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1. INTRODUCTION:

The problem of approximation arises in many contexts of "Numerical Analyses and Computing" [3, 4, 7 and 9]. Weirstrass [10] proved his celebrated approximation theorem: "…If, f ε C [a, b], then for every $\delta > 0$, ∃ a polynomial "p" such that "$\|f - p\| < \delta$". In other words, result established the existence of an algebraic polynomial in concerned variable capable of approximating the unknown function in that variable, as closely as we please! This result was a big beginning of the Mathematicians' interest in "Polynomial Approximation'' [1, 5, 6, and 9] of an unknown function using its values generated, experimentally or otherwise, at certain equidistant knots in the impugned interval of the relevant variable. The Great Russian mathematician Bernstein proved the Weirstrass theorem in a style, which was very thought-provoking and curious in many ways. He first noted a simple though a very significant feature of this theorem, namely that if it holds for C [0, 1], it does hold for C [a, b] and vice-versa. In fact, C [0, 1] and C [a, b] are essentially identical, for all practical purposes, inasmuch as they are linearly "isometric" as normed spaces, order isomorphic as lattices, and isomorphic as algebras (rings) [2]. Also, the most important contribution in the Bernstein's proof of the Weirstrass' theorem consisted in the fact that Bernstein actually displayed a sequence of polynomials that approximate a given function f ε C [0, 1]. If, f(x) is any bounded function on C [0, 1], the sequence of "Bernstein Polynomials" [1] for f(x) is defined by:

$$(B_n(f))(x) = \sum_{k=0}^{k=n} \binom{n}{k} x^k \cdot (1-x)^{(n-k)} \cdot f(k/n), \; x \in C[0,1]; \text{ Say, } E[f(x)] \tag{1.1}$$

The aim of the present paper, motivated by the "Iterative Statistical Bias-Reduction Intervention" in Sahai (2004) [8], is to propose such a variant of the aforesaid "Bernstein Polynomials" which would be much more amenable to that "Intervention"!

2. THE PROPOSITION OF THE DUAL-FUSION VARIANT OF THE BERNSTEIN POLYNOMIAL:

In context of the aforementioned sequence of "Bernstein Polynomials" for f(x), a significant observation which must be taken note of is that the use is made of the values of the unknown function "f (x)" at the equidistant-knots "k/n; k = 0(1) n", assumed to be knowable through the experiment(s) in the relevant scientific field of investigation or known otherwise. This fuller use of this aforementioned information about the "n+1" values of the unknown function "f (x)" (at the equidistant-knots "k/n; k = 0(1) n") is the "Key-Point" for the paper.

In any approximating polynomial operator use is made of the "Knots" and the corresponding "Weights". In our proposition of a variant of the "Bernstein's Polynomial" we propose to systematically introduce new corresponding

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weights, without changing the location of the "knots". We propose a "DUAL_FUSION" variant of the "Bernstein Polynomial" which is having a Systematic Bias, and is, therefore, more readily amenable to a Statistical "Bias-Reduction" strategy than is the original/ usual "Bernstein's Polynomial"!

We consider the following PRIMAL variant of the Bernstein's Polynomial:

$$\text{Say, } B^P (f; x) [n] = \sum_{k=0}^{k=n} \binom{n}{k} ((1+x)/2)^k \cdot ((1-x)/2)^{(n-k)} \cdot f (k/n) \tag{1.2}$$

The correspondingly **DUAL (-Weights)** variant of the Bernstein Polynomial would be:

$$\text{Say, } B^D (f; x) [n] = \sum_{k=0}^{k=n} \binom{n}{k} ((1-x)/2)^k \cdot ((1+x)/2)^{(n-k)} \cdot f (k/n) \tag{1.3}$$

We define the **"PRIMAL-DUAL-Fusion-Weights"** variant of the Bernstein Polynomial as follows:

$$\textbf{Say, PDFB (f; x) [n] = ((1 + x)/2). } \mathbf{B^P} \textbf{ (f; x) + ((1 – x)/2). } \mathbf{B^D} \textbf{ (f; x)} \tag{1.4}$$

To make comprehensible the systematicness of this (PRIMAL-DUAL Fusion) variant of the Bernstein Polynomial, say, PDFB (f; x) [n] we note that it will work for an approximation polynomial focusing interval [(1 - x)/2, (1 + x)/2] around "1/2", which will be [0, 1] for x = 1, and degenerating to the point "0" for x = 1. The impugned interval will be wider, the greater the value of "x"! For example, in the approximating polynomial in "x" for the values of x ε [0, 1]; the interval will be always centered around "0", symmetrically, e.g. [1/4, 3/4] for x =1/2. To balance the "Pull", systematically, the weights "((1 + x)/2)" and "((1 – x)/2)" are assigned to the relevant weights in B^P (f; x) [n] & B^D (f; x) [n]. These weights are also, respectively, "DUAL" to each-other, again! The aforesaid (PRIMAL-DUAL Fusion) variant of the Bernstein Polynomial, namely, PDFB (f; x) [n] will induce a "(Systematic)Bias" in the approximating "Polynomial" which is amenable, more systematically (than the original "Bernstein's Polynomial"), to the correction by "Statistical Bias-Reduction" 'Iteration(s)', to that extent which might please us.

3. THE ITERATIVE-BIAS REDUCTION-STRATEGY FOR THE PROPOSED DUAL-FUSION POLYNOMIAL:

We detail in this section, very briefly, the Iterative-Bias Reduction-Strategy for the Proposed Dual-Fusion Polynomial which is exactly analogous to that used in Sahai (2004) [8] for the Original "Bernstein's Polynomial". We note that the estimated values of the unknown function "f (x)" at the "knots: 'k/n'; k = 0 (1) n", as per the proposed "Dual Fusion Polynomial PDFB (f; x) [n]" would be:

$$\text{Say, Et PDFB (f; k/n) [n]} \tag{3.1}$$

Thus, the "Error of Estimation" @ 'the "knots: 'k/n'; k = 0 (1) n" would be:

$$\text{Say, Er PDFB (f; k/n) [n] = Et PDFB (f; k/n) [n] – f (k/n)} \tag{3.2}$$

We do note here that f (k/n)'s known for all "knots: 'k/n'; k = 0 (1) n".

Hence, using (3.2) and the proposed "Dual Fusion Polynomial PDFB (f; x) [n]", we get the "Error Polynomial" as:

$$\text{Say, Er PDFB (f; x) [n].} \tag{3.3}$$

The resultant "Bias-Reduced Polynomial" at "Iteration # 1" would then be:

$$\begin{aligned}\text{Say, I (1) UPDFB (f; x) [n]} &= \text{PDFB (f; x) [n] - Er PDFB (f; x) [n].}\\ &= \text{2.PDFB (f; x) [n] – PDFB}^2\text{ (f; x) [n]}\\ &= [\text{I} – (\text{I} – \text{PDFB})^2] \text{ (f; x) [n].}\end{aligned} \tag{3.4}$$

Similarly, the resultant "Bias-Reduced Polynomial" at "Iteration # 2" would then be:

Say, I (2) UPDFB (f; x) [n] = $[I - (I - PDFB)^3]$ (f; x) [n]. (3.5)

In general, "Bias-Reduced Polynomial" at "Iteration # J" would be:

Say, I (J) UPDFB (f; x) [n] = $[I - (I - PDFB)^{J+1}]$ (f; x) [n]. (3.6)

Apparently, "J" being the iteration #, has to be a positive integer!

Similar to what was noted in Sahai (2004) [8], in the absence of any conclusive analytical study [The derivable "Upper" bounds on the error of approximation (as noted in the paper by Sahai (2004) [8]) are not of much use as a smaller/ lower "Upper Bound" does not guarantee a better approximation and the extent of the resultant "GAIN" is unavailable, too!],we go for an empirical simulation study to illustrate the potential "GAIN" through our "Bias-Reduction Iteration(s)" on this proposed (PRIMAL-DUAL Fusion) variant of the Bernstein Polynomial, namely, PDFB (f; x) [n].

4. THE EMPIRICAL SIMULATION STUDY:

To illustrate the gain in efficiency by using our proposed Iterative Algorithm of Improvement of the proposed "Dual-Fusion" variant of the Bernstein Polynomial Approximation, we have carried an empirical study. We have taken the example-cases of n = 3, 4, and 5 (i.e. n + 1 = 4, 5, and 6 knots) in the empirical study to numerically illustrate the relative gain in efficiency in using the Algorithm vis-à-vis the Original "Dual-Fusion" variant of the Bernstein Polynomial proposed in each example case of the n-value. Essentially, the empirical study is a simulation one wherein we would have to assume that the function, being tried to be approximated, namely "f (x)" being known to us. Once again we have confined to the illustrations of the relative gain in efficiency by the Iterative Improvement for the following four illustrative functions:

f (x) = exp(x), ln(2+x), sin(2+x), and 10^x

To illustrate the POTENTIAL of improvement with our proposed Iterative Algorithm, we have considered THREE Iterations, and the numerical values of SEVEN quantities-three percentage relative errors (PREs) corresponding to our Improvement Iteration (# J = 1, or 2, or 3): PRE_I (J) UPDFB (f; x) [n], that to Original "Dual-Fusion" variant of the Bernstein Polynomial: PRE_PDFB (f; x) [n], and the corresponding Percentage Relative Gains (PRGs) in using our Iterative Algorithmic Reduced-Bias "Dual-Fusion" variant of the Bernstein Polynomial in place of the Original "Dual-Fusion" variant of the Bernstein Polynomial PRG_I (J) UPDFB (f; x) [n] (# J = 1, or 2, or 3). These quantities are defined as follows.

PRE_PDFB (f; x) [n] = $100.\ [\{ \int_0^1 \textbf{abs.}\,(\textbf{PDFB (f; x)[n] - f (x)})\, dx \} / \int_0^1 \textbf{f (x)}\, dx\ ];$ (4.1)

PRE_I (J) UPDFB (f; x) [n] = $100.\ [\{ \int_0^1 \textbf{abs.}\,(\textbf{I(J) UPDFB (f; x)[n] - f (x)})\, dx \} / \int_0^1 \textbf{f (x)}\, dx\ ];$ (4.2)

Wherein; J = 1 or 2 or 3. And, PRG_I (J) UPDFB (f; x) [n] =

$= 100.\ [\{\textbf{PRE\_PDFB (f; x)[n] - PRE\_I (J) UPDFB (f; x)[n]}\} / \{\textbf{PRE\_PDFB (f; x)[n]}\}]$ (4.3)

Wherein; J = 1 or 2 or 3.

The PREs respective to the Original Variant of Bernstein Polynomial and respective to the First, Second, and the Third Algorithmic Improvement Iteration Polynomials, respectively for each of the example # of approximation Knots/Intervals; and the PRGs (defined as above in (4.3)) by using the Proposed Algorithmic Improvement Iteration: I# (e.g. 1, or 2, or 3) Polynomials with the n intervals in [0, 1] over using the Original Variant of Bernstein Polynomial for the approximation of the (Targeted) function, 'f (x)' are tabulated in the APPENDIX in Tables 1–4.

5. CONCLUSION:

For all the FOUR illustrative functions, namely f (x) = exp (x); ln (2 +x); sin (2 + x), and 10^x, the PRGs are around 99% for n = 3, 4, and 5. It is very significant to note that the PRGs are above 99.5% for n=5, i.e. for only SIX 'Knots'!

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APPENDIX:

Table 1 :(Iterative) Algorithmic (In %) Relative (Absolute) Efficiency/ Gain for f (x) = exp(x).

Items ↓	**n→ 3**	**4**	**5**
PRE_PDFB (f; x) [n]	19.34918909	18.62091287	18.18048680
PRE_I (1) UPDFB (f; x) [n]	0.11000302	0.10770714	0.10664681
PRE_I (2) UPDFB f; x) [n]	0.06279666	0.06595319	0.06816574
PRE_I (3) UPDFB (f; x) [n]	0.06068674	0.06317454	0.06521435
PRG_I (1) UPDFB (f; x) [n]	99.43148512	99.42157965	99.41339959
PRG_I (2) UPDFB (f; x) [n]	99.67545586	99.64581119	99.62506092
PRG_I (3) UPDFB (f; x) [n]	99.68636029	99.66073339	99.64129481

Table 2: (Iterative) Algorithmic (In %) Relative (Absolute) Efficiency/Gain for f (x) = ln (2+x).

Items ↓	**n→ 3**	**4**	**5**
PRE_PDFB (f; x) [n]	6.91836557	7.03802410	7.10923650
PRE_I (1) UPDFB (f; x) [n]	0.02088038	0.02495295	0.04030345
PRE_I (2) UPDFB f; x) [n]	0.00686818	0.01276364	0.03716380
PRE_I (3) UPDFB (f; x) [n]	0.00220013	0.00896998	0.03666696
PRG_I (1) UPDFB (f; x) [n]	99.69818911	99.64545514	99.43308324
PRG_I (2) UPDFB (f; x) [n]	99.90072527	99.81864731	99.47724616
PRG_I (3) UPDFB (f; x) [n]	99.96819856	99.87254963	99.48423477

Table 3: (Iterative) Algorithmic (In %) Relative (Absolute) Efficiency/Gain for f (x) = sin (2+x).

Items ↓	**n→ 3**	**4**	**5**
PRE_PDFB (f; x) [n]	25.21042851	24.56619480	24.18447530
PRE_I (1) UPDFB (f; x) [n]	0.05467121	0.05495187	0.05515142
PRE_I (2) UPDFB f; x) [n]	0.03952980	0.03999141	0.04035494
PRE_I (3) UPDFB (f; x) [n]	0.03737522	0.03725905	0.03737931
PRG_I (1) UPDFB (f; x) [n]	99.78314046	99.77631102	99.77195523
PRG_I (2) UPDFB (f; x) [n]	99.84320060	99.83720959	99.83313700
PRG_I (3) UPDFB (f; x) [n]	99.85174699	99.84833198	99.84544089

Table 4: (Iterative) Algorithmic (In %) Relative (Absolute) Efficiency/Gain for f (x) = 10^x.

Items ↓	**n→ 3**	**4**	**5**
PRE_PDFB (f; x) [n]	51.54498557	47.59919214	45.19394949
PRE_I (1) UPDFB (f; x) [n]	0.65678416	0.54509289	0.48224711
PRE_I (2) UPDFB f; x) [n]	0.20503808	0.14011181	0.10846483
PRE_I (3) UPDFB (f; x) [n]	0.05530318	0.01902893	0.00629602
PRG_I (1) UPDFB (f; x) [n]	98.72580397	98.85482743	98.93293878
PRG_I (2) UPDFB (f; x) [n]	99.60221527	99.70564246	99.76000141
PRG_I (3) UPDFB (f; x) [n]	99.89270890	99.96002256	99.98606889