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OPTIMAL STRATEGY ANALYSIS OF AN N-POLICY TWO-PHASE $M^X/E_K/1$ GATED QUEUEING SYSTEM WITH SERVER STARTUP AND BREAK DOWNS

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ABSTRACT

This paper deals with the optimal operation of a single removable server in an $M^X/E_k/1$ two-phase queueing system with gating, server startup and unpredictable breakdowns under N-policy. Arrivals occur in batches of random size X according to a compound Poisson process and waiting customers receive batch service all at a time in the first phase and proceed to the second phase to receive individual service. After providing the second phase of service, the server returns to the first phase to see if new customers have arrived. If there are any waiting customers, the server starts the cycle by providing them batch service followed by individual service. If no customer is waiting, the server leaves for a random period of vacation. As soon as the queue length reaches a threshold value N ($N \geq 1$) the server is turned on and is temporarily unavailable to serve the waiting batch of customers. Because, when it comes back from vacation it needs a startup time before the batch service commences in the first phase. While the server is working with the second phase of individual service, it may breakdown at any instant and is immediately repaired. In the first phase of service, the batch includes only those customers who are already in the queue. This criterion is called gating. Explicit expressions for the steady state distribution of the number of customers in the system and hence the expected system length is derived. The total expected cost function is developed to determine the optimal threshold of N at a minimum cost. Numerical experiment is performed to validate the analytical results. The sensitivity analysis has been carried out to examine the effect of different parameters in the system.

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Key words: *Vacation, N-policy, Queueing System, Two-phase, Startup, Breakdowns.*

INTRODUCTION

We investigate the optimal control policy of a removable and un-reliable server for an $M^X/E_k/1$ two-phase gated queueing system with server startups, where the removable server operates an N-policy and takes an exponentially distributed vacation. Removable server means, the server can be turned on or off depending on whether customers are waiting in the system or no customer is in the system. The N-policy means that the server is turned on when N ($N \geq 1$) or more customers are present, and off when the system is empty. An un-reliable server means that the server is subject to unpredictable breakdowns.

In this system, it is assumed that customers arrive following a compound Poisson process where the arrival rate is a random variable. The service is in two-phases, where the first phase of service is batch service to all customers waiting in the queue and the second phase of service is individual to each customer in the batch and consists of *k* exponential phases. On completion of the second phase of service, the server will go back to the first phase. If the customers are waiting in the queue it will provide batch service followed by individual service. Otherwise, it will proceed to a vacation of random length and is turned on as and when the queue length reaches or exceeds N. After coming back from a vacation the server needs a startup time for preparatory work before the batch service. While the server is working in the second phase, it is assumed that the server may breakdown at any time, and if the server fails, it is immediately sent for repair.

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Such a control model frequently occurs in the area of computer networking, communication and in flexible production (manufacturing) systems. Let us consider a production process where the machine producing certain items may require two phases of service, such as preliminary checking followed by the usual process to complete the processing of the raw materials. After these two phases of service, the production process either needs to be stopped if there are no jobs on hand or may continue the processing of the raw materials. Once the production process is stopped it may need some startup time for re-starting the process. There may be breakdowns during the production process and needs to be repaired immediately.

Krishna and Lee [11] and Doshi [7] studied distributed systems where all the customers waiting in the queue receive batch service in the first phase followed by individual service to every customer in the second phase. Selvam and Sivasankaran [14] introduced the two-phase queueing system with server vacations. Kim and Chae [10] investigated the two-phase queueing system with N-policy.

Baker [1] first proposed the N- policy M/M/1 queueing system with server startups. Later, Borthakur et al. [2] extended the Baker's result to the general startup time. Minh [13] first studied the N-policy M/G/1 queueing system with exponential startup times. Medhi and Templeton [12] extended Minh's result to the general startup times.

Several authors have investigated queueing models with server breakdowns and vacations in different frame works in the recent past. Wang [18] for the first time proposed Markovian queueing system under the N-policy with server breakdowns. Wang [19] Wang et al.[20] and Wang et al.[22] extended the model proposed by Wang[18] to $M/E_K/1$, $M/H_2/1$ and $M/H_K/1$ queueing systems respectively. Ke[8] presented the control policy of a removable and unreliable server, for an $M^X/M/1$ queueing system, where the removable server operates an N-policy. Wang et al.[25] presented the optimal control of the N-policy M/G/1 queueing system with a single removable and unreliable server. Ke and Lin[9] studied the $M^X/G/1$ queueing system under N-policy with an unreliable server with single vacation. Wang and Huang[24] used a maximum entropy approach to study a single removable and unreliable server in an M/G/1 queue operating under the (p,N) – policy. Tadj and Choudhury[15] analyzed a bulk service quorum queueing system with an unreliable server, Poisson input and general service and repair times. Choudhury and Madan[4] presented the optimal policy for a two-stage batch arrival queueing system with a modified Bernoulli schedule vacation under N-policy.Vasanta Kumar and Chandan [16],Vasanta Kumar et al [17] presented the optimal operating policy for a two-phase $M^X/E_K/1$ queueing system under N-policy with and without breakdowns .Wang[21] considered an M/G/1 model with an additional second phase of optional service with an assumption that the server is subjected to breakdowns and repairs in which it is assumed that the second optional service times follow an exponential distribution. Choudhury and Deka[3] generalized this model by introducing the concept of repeated attempts. Choudhury and Tadj [5] generalized this model by introducing the concept of delayed repair. Later, Choudhury et al.[6] investigated such a type of model, where concept of N-policy is also investigated along with a delayed repair for batch arrival queueing system.

Existing research works for the N-policy, including those mentioned above, have never investigated cases involving both server startups and breakdowns for the bulk arrival two-phase queueing systems.

The four main objectives for which the analysis has been carried out in this paper for the optimal control policy are:

(i) To establish state equations to obtain the steady state probability distribution of the number of customers in the system.
(ii) To derive an expression for the expected number of customers in the system.
(iii) To formulate the expected cost function for the system, and determine the optimum value of the control parameter N.
(iv) To carryout a sensitivity analysis of the optimal value of N and the minimum expected cost for various system parameters through numerical illustrations.

2. THE SYSTEM AND ASSUMPTIONS

In the fitness of realistic situation it is more appropriate to consider the arrival process is a compound Poisson process (with arrival rate λ) of independent and identically distributed random batches of customers where each batch size X, has a probability density function $\{a_n : a_n = P(X = n),\ n \geq 1\}$. The first phase of service is a batch service to all the customers waiting in the queue and the batch service time is assumed to be exponentially distributed with mean $1/\beta$. On completion of the batch service, the server provides individual service to all customers in the batch in FIFS order. The individual service is in k independent and identically distributed exponential phases with mean $1/k\mu$. To simplify the analysis the probability sequence $\{c_{nk}\}$ is used to represent the number of arrival phase, i.e., if the arriving group has size 'n', then it has 'nk' phases with probability c_{nk}. After the completion of individual service, the server returns to the first phase to serve the customers who have arrived. If the customers are waiting, the server restarts the cycle by providing batch service followed by individual service. If no customer is waiting, the server takes a vacating and returns from the vacation only after N customers accumulate in the queue and spend a random time 't' for pre-service, which is assumed to follow an exponential distribution with mean $1/\theta$. As soon as the startup period is over, the server begins batch service. Arrivals during the batch service are not served in the same batch, but are served in the next visit of the server to the batch queue. While serving in the individual queue, the server may fail at any time with a Poisson breakdown rate α. When the server fails, it is immediately repaired, where the repair times are exponentially distributed with mean $1/\gamma$. After repair the server immediately resumes individual service. Further, we assume that input process, startup time, server's life time, server's repair time and service time random variables are mutually independent of each other.

A diagram showing the transition among vacation, startup, batch service, individual service and breakdown periods is given in Figure 1.

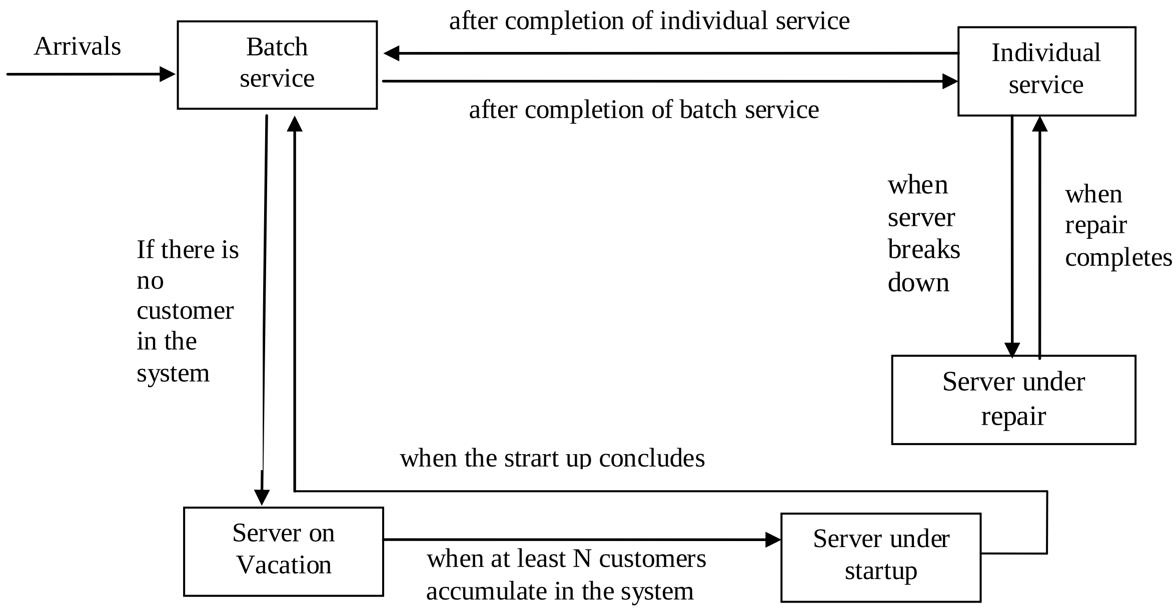

Figure 1. Transition diagram

3. STEADY – STATE ANALYSIS

In steady-state the following notations are used.

$P_{0,\,i,\,0}$ = The probability that there are i service phases in batch queue when the server is on vacation, where i=0, k, 2k, 3k,. . . ,(N-1) k.

$P_{1,\,i,\,0}$ =The probability that there are i service phases in batch queue when the server is doing pre-service (startup work), where i=Nk, (N+1)k, (N+2)k, ...

$P_{2,\,i,\,0}$ =The probability that there are i service phases in batch queue when the server is in batch service, where i= k, 2k, 3k . . .

$P_{3,\,i,\,j}$ = The probability that there are i service phases in batch queue and j service phases in individual queue when the server is in individual service, where i = 0, k, 2k, 3k, . . . and j = 1, 2, 3, ...

$P_{4,\,i,\,j}$ =The probability that there are i service phases in batch queue and j service phases in individual queue when the server is in individual service but found to be broken down, where i = 0, k, 2k, 3k,…and j = 1, 2, 3, ...

The steady-state equations satisfied by the system size probabilities are as follows:

$$\lambda\ P_{0,0,0} = k\mu\ P_{3,0,1} \tag{1}$$

$$\lambda\ P_{0,i,0} = \lambda \sum_{l=k}^{i} c_l\ P_{0,\,i-l,\,0},\quad i= k, 2k, 3k, \ldots, (N-1)\,k. \tag{2}$$

$$(\lambda+\theta)\ P_{1,\,Nk,\,0} = \lambda \sum_{l=k}^{NK} c_l\, P_{0,\,Nk-l,\,0}. \tag{3}$$

$$(\lambda+\theta)\, P_{1,i,0} = \lambda \sum_{l=k}^{i-NK} c_l\, P_{1,i-l,0} + \lambda \sum_{l=i-(N-1)k}^{i} c_l\, P_{0,i-l,0}, \quad i = (N+1)k, (N+2)k, (N+3)k, \ldots \tag{4}$$

$$\beta\, P_{2,i,0} = k\mu\, P_{3,i,1}, \quad i = k, 2k, 3k, \ldots, (N-1)k. \tag{5}$$

$$\beta\, P_{2,i,0} = k\mu\, P_{3,i,1} + \theta\, P_{1,i,0}, \quad i = Nk, (N+1)k, (N+2)k, \ldots. \tag{6}$$

$$\lambda+\alpha+k\mu)P_{3,0,j} = k\mu\, P_{3,0,j+1} + \pi_0\, \beta\, P_{2,j,0} + \gamma\, P_{4,0,j}, \quad j \geq 1. \tag{7}$$

$$(\lambda+\alpha+k\mu)P_{3,i,j} = k\mu\, P_{3,i,j+1} + \pi_{i/k}\, \beta\, P_{2,j,0} + \lambda \sum_{l=k}^{i} c_l\, P_{3,i-l,j} + \gamma\, P_{4,i,j}, \; i = k, 2k, 3k, \ldots, \; j \geq 1. \tag{8}$$

$$(\lambda+\gamma)\, P_{4,0,j} = \alpha\, P_{3,0,j}, \quad j \geq 1 \tag{9}$$

$$(\lambda+\gamma)\, P_{4,i,j} = \alpha\, P_{3,i,j} + \lambda \sum_{l=k}^{i} c_l\, P_{4,i-l,j}, \; i = k, 2k, 3k, \ldots, \; j \geq 1. \tag{10}$$

where π_i is the probability that *i* service phases arrive during sojourn of the server at the batch queue.

The following probability generating functions are defined

$$G_0(z) = \sum_{i=0}^{(N-1)k} P_{0,i,0}\, z^i, \quad G_1(z) = \sum_{i=Nk}^{\infty} P_{1,i,0}\, z^i, \; G_2(z) = \sum_{i=k}^{\infty} P_{2,i,0}\, z^i,$$

$$G_3(z,y) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} P_{3,i,j}\, z^i y^j, \; G_4(z,y) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} P_{4,i,j}\, z^i y^j, \; R_j(z) = \sum_{i=0}^{\infty} P_{3,i,j}\, z^i,$$

$$S_j(z) = \sum_{i=0}^{\infty} P_{4,i,j}\, z^i, \; C(z) = \sum_{i=1}^{\infty} c_i\, z^i, \quad A(z) = \sum_{i=1}^{\infty} a_i\, z^i, \text{ and } \pi(z) = \sum_{i=0}^{\infty} \pi_{i/k}\, z^{i/k},$$

where $|z| \leq 1$ and $|y| \leq 1$.

C(z) is the probability generating function of the number of phases in the batch and A(z) is the probability generating function of the number of customers in the system. It is found that $E(C) = C^1(1)$ and $E(C(C-1)) = C^{11}(1)$.

It can be shown that $C^1(1) = kA^1(1)$ and $C^{11}(1) = k^2 A^{11}(1) + k(k-1)A^1(1)$.

Also $\pi_{i/k} = \dfrac{(\lambda C'(1))^{i/k}\, \beta}{(\lambda C'(1)+\beta)^{i/k+1}}$, $\pi'(1) = \dfrac{\lambda C'(1)}{\beta}$ and $\pi''(1) = \dfrac{2(\lambda C'(1))^2}{\beta^2}$.

Using equation (2), we get $\quad P_{0,i,0} = y_i\, P_{0,0,0}$, $\quad i = k, 2k, 3k, \ldots, (N-1)k$,

where y_i's are defined as $y_0 = 1$ and $y_i = \sum_{l=k}^{i} c_l\, y_{i-l}$.

Hence, $G_0(z) = \sum_{i=0}^{(N-1)k} P_{0,i,0}\, z^i = P_{0,0,0}\, y_N(z)$,

$$\text{where} \quad y_N(z) = \sum_{i=0}^{(N-1)k} y_i z^i, \; y_N(1) = \sum_{i=0}^{(N-1)k} y_i, \text{ and } y_N^1(1) = \sum_{i=1}^{(N-1)k} i y_i. \tag{11}$$

Solving equations (3) to (10) using generating functions, we get

$$[\lambda(1-C(z)) + \theta]\, G_1(z) = \lambda\, P_{0,0,0} + [\lambda(C(z)-1)]\, G_0(z). \tag{12}$$

$\beta G_2(z) = k\mu R_1(z) + \theta G_1(z) - \lambda P_{0,0,0}$. (13)

$[\lambda y(1-C(z)) + \alpha y + k\mu(y-1)] G_3(z, y) = \gamma y G_4(z, y) + \beta y \pi(z) G_2(y) - k\mu y R_1(z)$. (14)

$[\lambda(1-C(z)+\gamma]G_4(z,y) = \alpha G_3(z,y)$ (15)

The total probability generating function G(z, y) is given by

$G(z, y) = G_0(z) + G_1(z) + G_2(z) + G_3(z, y) + G_4(z, y)$.

The normalizing condition is

$G(1, 1) = G_0(1) + G_1(1) + G_2(1) + G_3(1,1) + G_4(1, 1)$. (16)

From equations (11) to (15)

$G_0(1) = y_N(1)P_{0,0,0}$, (17)

$G_1(1) = (\lambda/\theta)\, P_{0,0,0}$, (18)

$G_2(1) = k\mu\, (R_1(1)/\beta)$, (19)

$$G_3(1,1) = \frac{\left(\theta\beta G_1^1(1) + \lambda k\mu C^1(1) R_1(1)\right)\gamma}{\beta[k\mu\gamma - \lambda C^1(1)(\alpha+\gamma)]} \quad (20)$$

and $G_4(1,1) = (\alpha/\gamma)\, G_3(1,1)$. (21)

Normalizing condition (16) yields

$R_1(1) = \lambda C^1(1)/(k^2\mu)$. (22)

Substituting the value of $R_1(1)$ in (20) and (21) yields

$G_3(1, 1) = \lambda C^1(1)/(k\mu)$ and $G_4(1, 1) = (\alpha/\gamma)[\lambda C^1(1)/(k\mu)]$.

Under steady state conditions, let P_0, P_1, P_2, P_3 and P_4 be the probabilities that the server is in vacation, startup, in batch service, in individual service and breakdown states respectively. Then,

$P_0 = G_0(1) = y_N(1)P_{0,0,0}$, (23)

$P_1 = G_1(1) = \lambda P_{0,0,0}/\theta$, (24)

$P_2 = G_2(1) = \lambda C^1(1)/(k\beta)$, (25)

$P_3 = G_3(1, 1) = \lambda C^1(1)/(k\mu)$, (26)

and $P_4 = G_4(1,1) = (\alpha/\gamma)[\lambda C^1(1)/(k\mu)]$, (27)

where $$P_{0,0,0} = \left[1 - \left(\frac{\lambda}{\mu}\left(1+\frac{\alpha}{\gamma}\right) + \frac{\lambda}{\beta}\right)\frac{C^1(1)}{k}\right]\frac{\theta}{(\lambda + \theta y_N(1))}.$$

4. EXPECTED NUMBER OF CUSTOMERS IN THE SYSTEM

Using the probability generating functions expected number of customers in the system at different states are presented below.

Let L_0, L_1, L_2, L_3 and L_4 be the expected number of customers in the system when the server is in vacation, in startup, in batch service, in individual service and breakdown states respectively.

$$\text{Then}\quad L_0 = \sum_{i=0}^{(N-1)k} i\,P_{0,i,0} = G_o^1(1) = y_N^1(1)P_{0,0,0}, \tag{28}$$

$$L_1 = \sum_{i=Nk}^{\infty} i\,P_{1,i,0} = G_1^1(1) = \frac{\lambda(\lambda+\theta\,y_N(1))\,C^1(1)}{\theta^2}\,P_{0,0,0}, \tag{29}$$

$$L_2 = \sum_{i=k}^{\infty} i\,P_{2,i,0} = G_2^1(1) = \lambda\,C^1(1)\,/\beta, \tag{30}$$

$$L_3 = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}(i+j)\,P_{3,i,j} = G_3^1(1,1)$$

$$= \rho\left\{\frac{1}{1-\rho_1} + \frac{\lambda C^1(1)(\lambda+\theta y_N(1))}{\theta^2(1-\rho_1)}P_{0,0,0} + \frac{\lambda\rho\alpha C^1(1)}{\gamma^2(1-\rho_1)} + \frac{y_N^1(1)}{(1-\rho_1)}P_{0,0,0} + \frac{\lambda C^1(1)}{\beta(1-\rho_1)} + \frac{C^{11}(1)}{2(1-\rho_1)C^1(1)} + \frac{(\lambda C^1(1))^2}{k\beta^2(1-\rho_1)} - \frac{\lambda C^{11}(1)}{2k\beta(1-\rho_1)}\right\} \tag{31}$$

$$\text{and}\quad L_4 = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}(i+j)\,P_{4,i,j} = G_4^1(1,1) = \frac{\alpha}{\gamma}\left[G_3^1(1,1) + \frac{\lambda\rho C^1(1)}{\gamma}\right], \tag{32}$$

where $\rho = \lambda\,C^1(1)/(k\,\mu)$, $\rho_1 = \rho\ (1 + \frac{\alpha}{\gamma})$

and $P_{0,0,0} = \left[1-\rho_1 - \frac{\lambda C^1(1)}{k\beta}\right]\frac{\theta}{(\lambda+\theta y_N(1))}$.

The expected number of phases in the system is given by

$$L(P) = L_0 + L_1 + L_2 + L_3 + L_4$$

$$= \frac{\rho_1}{1-\rho_1} + \frac{y_N^1(1)}{(1-\rho_1)}P_{0,0,0} + \frac{\lambda(\lambda+\theta y_N(1))C^1(1)}{\theta^2(1-\rho_1)}P_{0,0,0} + \frac{\lambda C^1(1)\alpha\rho}{\gamma^2(1-\rho_1)} + \frac{\lambda C^1(1)}{\beta(1-\rho_1)} + \frac{\rho_1 C^{11}(1)}{2(1-\rho_1)C^1(1)} + \frac{(\lambda C^1(1))^2\rho_1}{k\beta^2(1-\rho_1)} - \frac{\lambda C^{11}(1)\rho_1}{2k\beta(1-\rho_1)}. \tag{33}$$

Expressing L (P) in terms of $A^1(1)$ and $A^{11}(1)$

$$L(P) = \frac{\rho_1}{1-\rho_1} + \frac{y_N^1(1)}{(1-\rho_1)}P_{0,0,0} + \frac{\lambda(\lambda+\theta y_N(1))kA^1(1)}{\theta^2(1-\rho_1)}P_{0,0,0} + \frac{\lambda kA^1(1)\alpha\rho}{\gamma^2(1-\rho_1)} + \frac{\lambda kA^1(1)}{\beta(1-\rho_1)} + \frac{\rho_1(KA^{11}(1)+(k-1)A^1(1))}{2(1-\rho_1)A^1(1)} + \frac{(\lambda kA^1(1))^2\rho_1}{k\beta^2(1-\rho_1)} - \frac{\lambda(kA^{11}(1)+(k-1)A^1(1))\rho_1}{2\beta(1-\rho_1)}. \tag{34}$$

where $\rho = \dfrac{\lambda A^1(1)}{\mu}$, $\rho_1 = \rho\left(1+\dfrac{\alpha}{\gamma}\right)$ and $P_{0,0,0} = \left[1-\rho_1-\left(\dfrac{\lambda A^1(1)}{\beta}\right)\right]\dfrac{\theta}{(\lambda+\theta y_N(1))}$.

Then, the expected number of customers in the system is given by

$$L(N)=\frac{1}{k}\left[L(P)-\left(\frac{k+1}{2}\right)\left(\frac{\lambda}{\mu}\left(1+\frac{\alpha}{\gamma}\right)+\frac{\lambda}{\beta}\right)A^1(1)\right]+\left(\frac{\lambda}{\mu}\left(1+\frac{\alpha}{\gamma}\right)+\frac{\lambda}{\beta}\right)A^1(1). \quad (35)$$

5. SOME OTHER SYSTEM CHARACTERISTICS

Let E_0, E_1, E_2, E_3 and E_4 denote the expected length of idle period, startup period, batch service period, individual service period and breakdown period respectively. Then the expected length of a cycle is given by

$$E_C = E_0 + E_1 + E_2 + E_3 + E_4. \quad (36)$$

The long-run fraction of time, the server is idle, in startup, in batch service, in individual service and breakdown states are respectively given b

$$\frac{E_0}{E_C} = P_0 = \frac{y_N(1)(1-\rho_1-\lambda A^1(1)/\beta)}{(\lambda/\theta+y_N(1))}, \quad (37)$$

$$\frac{E_1}{E_C} = P_1 = \frac{(\lambda/\theta)(1-\rho_1-\lambda A^1(1)/\beta)}{(\lambda/\theta+y_N(1))}, \quad (38)$$

$$\frac{E_2}{E_C} = P_2 = \lambda A^1(1)/\beta, \quad (39)$$

$$\frac{E_3}{E_C} = P_3 = \rho \quad (40)$$

and $\frac{E_4}{E_C} = P_4 = G_4(1) = \rho\ (\alpha/\gamma)$. (41)

Expected length of idle period = $E_0 = y_N(1)/\lambda$. Substituting this in equation (37) yields

$$E_C = \frac{(\lambda/\theta)+y_N(1)}{\lambda\left(1-\rho_1-\lambda A^1(1)/\beta\right)}. \quad (42)$$

6. OPTIMAL COST STRUCTURE

We develop a steady state total expected cost function per unit time for the N-policy two-phase $M^X/E_K/1$ gated queueing system with startup and server break downs, in which N is a decision variable. With the cost structure being considered, the objective is to determine the optimal N-policy, so as to minimize this function.

Let
C_h = holding cost per unit time for each customer present in the system,
C_0 = cost per unit time for keeping the server on and in operation,
C_m = startup cost per unit time per cycle,
C_s = setup cost per cycle,
C_b = breakdown cost per unit time for the unreliable server and
C_r = reward per unit time for the server being in vacation.

The average cost per unit time is given by

$$T(N) = C_h L(N) + C_0\left(\frac{E_2+E_3}{E_c}\right) + C_m\left(\frac{E_1}{E_c}\right) + C_b\left(\frac{E_4}{E_c}\right) + C_s\left(\frac{1}{E_c}\right) - C_r\left(\frac{E_0}{E_c}\right). \quad (43)$$

From equations (34) to (40), it is observed that $\frac{E_2}{E_c}$, $\frac{E_3}{E_c}$ and $\frac{E_4}{E_c}$ are not functions of decision variable N. Hence for determination of the optimal operating N-policy, minimizing T(N) in equation (43) is equivalent to minimizing

$$T_1(N) = C_h L(N) + C_m\left(\frac{E_1}{E_c}\right) + C_s\left(\frac{1}{E_c}\right) - C_r\left(\frac{E_0}{E_c}\right)$$

$$= \frac{\left[1-\rho_1 - \lambda A^1(1)/\beta\right]}{\left(y_N(1)+\lambda/\theta\right)(1-\rho_1)}\left[\frac{C_h}{k} y_N^1(1) + (1-\rho_1)\left\{C_m\left(\lambda/\theta\right) + \lambda C_s - C_r\, y_N(1)\right\}\right]. \quad (44)$$

It is hard to prove that $T_1(N)$ is convex. But we now present a procedure that makes it possible to calculate the optimal threshold N^*.

RESULT

Under the long-run expected average cost criterion, the optimal threshold N^* for the model is given by

$$N^* = \min\left\{ p \ge 1 / \sum_{i=0}^{p} (p-i)\, y_i + \frac{p\lambda}{\theta} > \frac{\lambda(1-\rho_1)k}{C_h}\left(\frac{C_m - C_r}{\theta} + C_s\right)\right\}$$

Proof:

Let $T = (1-\rho_1 - \lambda A'(1)/\beta)/(1-\rho_1)$,

$J_p = \sum_{i=0}^{p} y_i + \frac{\lambda}{\theta}$ and $M_p = \sum_{i=1}^{p} i\, y_i$, where p = 1, 2, 3, ...

Then we have

$$T_1(p+1) - T_1(p) = \frac{C_h T}{k}\left(\frac{M_p}{J_p} - \frac{M_{p-1}}{J_{p-1}}\right) + (1-\rho_1)T\left[C_m\frac{\lambda}{\theta} + \lambda C_s\left(\frac{1}{J_p} - \frac{1}{J_{p-1}}\right) - C_r\left(\frac{J_p - \lambda/\theta}{J_p} - \frac{J_{p-1}-\lambda/\theta}{J_{p-1}}\right)\right]$$

$$= \frac{T\, y_p}{J_p J_{p-1}} h(p),$$

where $h(p) = \frac{C_h}{k}\left(pJ_p - M_p\right) - \lambda\left(1-\rho_1\right)\left(\frac{C_m - C_r}{\theta} + C_s\right).$

By definition of J_p, M_p and T,

$\frac{C_h}{k}\left(pJ_p - M_p\right) > 0$ and $\frac{T\, y_p}{J_p J_{p-1}} > 0$.

Thus the sign of h (p) determines whether $T_1(p)$ increases or decreases.

Let q be the smallest p such that h(p)>0. Then

$$h(q+1) = \frac{C_h}{k}\left((q+1)J_{q+1} - M_{q+1}\right) - \lambda(1-\rho_1)\left(\frac{C_m - C_r}{\theta} + C_s\right)$$
$$= \frac{C_h}{k}\left(qJ_q - M_q\right) - \lambda(1-\rho_1)\left(\frac{C_m - C_r}{\theta} + C_s\right) + \frac{C_h}{k} M_q$$
$$= h(q) + \frac{C_h}{k} M_q > 0.$$

Thus, $T_1(p) > T_1(q)$ for every p>q.

Hence, $N^* =$ first p such that $h(p) > 0$

$$= \min\left\{ p \geq 1 / \sum_{i=0}^{p} (p-i)\,y_i + \frac{p\lambda}{\theta} > \frac{\lambda(1-\rho_1)k}{C_h}\left(\frac{C_m - C_r}{\theta} + C_s\right)\right\}.$$

7. NUMERICAL ILLUSTRATION

In this section we present some examples to illustrate how the cost structure can be utilized to make the decision regarding the optimal value N* to minimize the average cost for different values of various parameters. We assume that the batch size X follows a geometric distribution with parameter *p*.

Then, $A^1(1) = 1/p$, $A^{11}(1) = 2(1-p)/p^2$ and the expected number of customers in the system is given by

$$L(N) = \frac{1}{k}\left[\frac{\rho_1}{1-\rho_1} + \frac{y_N^1(1)}{1-\rho_1}P_{0,0,0} + \frac{\lambda(\lambda+\theta y_N(1))k}{\theta^2(1-\rho_1)p}P_{0,0,0} + \frac{\alpha\lambda k\rho}{\gamma^2(1-\rho_1)p} + \frac{\lambda k}{p\beta(1-\rho_1)} + \frac{\lambda^2 k\rho_1}{\beta^2(1-\rho_1)p^2}\right.$$
$$\left. + \frac{\rho_1(2k-(k+1)p)}{2(1-\rho_1)p} - \frac{\lambda(2k-(k+1)p)\rho_1}{2\beta(1-\rho_1)p^2} - \left(\frac{k+1}{2}\right)\left(\frac{\lambda}{\beta} + \frac{\lambda}{\mu}\left(1+\frac{\alpha}{\gamma}\right)\right)\frac{1}{p}\right] + \left\{\frac{\lambda}{\beta} + \frac{\lambda}{\mu}\left(1+\frac{\alpha}{\gamma}\right)\right\}\frac{1}{p},$$

where $y_i = \sum_{l=1}^{i} a_l y_{i-l}$, $y_0 = 1$, $a_l = p(1-p)^{l-1}$, $\rho = \frac{\lambda}{p\mu}$, $\rho_1 = \rho\left(1+\frac{\alpha}{\gamma}\right)$,

$P_{0,0,0} = \left[1-\rho_1-\frac{\lambda}{p\beta}\right]\frac{\theta}{(\lambda+\theta y_N(1))}$ and $1/p$ is the mean size of the arrival batch.

For convenience of computations, the following cost elements are considered.

Case 1: $C_h = 5$, $C_o = 100$, $C_m = 300$, $C_b = 125$, $C_r = 25$, $C_s = 500$

Case 2: $C_h = 5$, $C_o = 200$, $C_m = 500$, $C_b = 250$, $C_r = 50$, $C_s = 1250$

Case 3: $C_h = 5$, $C_o = 400$, $C_m = 800$, $C_b = 500$, $C_r = 100$, $C_s = 2500$

Case 4: $C_h = 10$, $C_o = 400$, $C_m = 800$, $C_b = 500$, $C_r = 100$, $C_s = 2500$

Case 5: $C_h = 50$, $C_o = 400$, $C_m = 800$, $C_b = 500$, $C_r = 200$, $C_s = 2500$

The joint optimum threshold value N* and its minimum expected cost $T(N^*)$ for the above five cost cases are shown in Table 1 for $(\mu,\beta,\gamma,\alpha,p,k) = (10,5,4,0.1,0.5,3)$ and for various values of (λ,θ). We observe from Table 1 that (i) N^* and $T(N^*)$ increase as λ increases for any case, (ii) N^* remains constant and T(N*) decreases for increase in the values of θ from 2 to 5.

Table 1. The effect of (λ,θ) on the optimal threshold value N* and the minimum expected cost for the five cost cases $(\mu,\beta,\gamma,\alpha,p,k) = (10,5,0.4,0.1,0.5,3)$.

	(λ,θ)	(0.4, 2)	(0.6, 2)	(1.8, 2)	(1.0, 2)	(0.6, 2)	(0.6, 3)	(0.6, 4)	(0.6, 5)
Case 1	N^* $T(N^*)$	22 41.23	27 60.52	30 76.49	32 89.72	26 60.54	26 58.60	26 57.63	26 57.04
Case 2	N^* $T(N^*)$	35 66.15	41 102.61	46 133.97	50 161.09	41 102.61	41 100.54	41 99.50	41 98.88
Case 3	N^* $T(N^*)$	49 101.57	58 171.79	66 234.86	71 292.00	57 171.79	57 169.45	57 168.26	57 167.55
Case 4	N^* $T(N^*)$	34 130.65	36 204.28	46 266.33	50 320.83	41 203.50	41 199.93	41 198.14	41 197.06
Case 5	N^* $T(N^*)$	15 183.47	18 286.24	20 366.96	21 429.71	18 286.24	18 274.89	18 269.19	18 265.45

The optimum value of N, N^* and its minimum expected cost $T(N^*)$ for the five cost cases are shown in Table 2 for $(\lambda,\theta,\gamma,\alpha,p,k)=(0.6,2,0.4,0.1,0.5,3)$ and for various values of (μ,β). We observe from Table 2 that (i) $T(N^*)$ decreases as μ and β increases for any case, (ii) N^* shows increasing trend as μ increases from 7 to 13 and (iii) N* does not change as β increases from 4 to 7.

Table 2 The effect of (μ,β) on the optimal threshold value N* and the minimum expected cost for the five cost cases $(\lambda,\theta,\gamma,\alpha,p,k)=(0.6,2,0.4,0.1,0.5,3)$.

	(μ,β)	(7, 5)	(9, 5)	(11, 5)	(13, 5)	(10, 4)	(10, 5)	(10,6)	(10,7)
Case 1	N^* $T(N^*)$	26 67.71	27 62.38	27 59.00	28 56.66	27 64.71	27 60.52	27 57.73	27 55.74
Case 2	N^* $T(N^*)$	40 116.93	41 106.32	42 99.56	42 94.88	42 112.56	42 102.61	42 95.97	42 91.23
Case 3	N^* $T(N^*)$	56 201.42	58 179.48	59 165.48	60 155.78	58 194.76	58 171.79	58 156.48	58 145.54
Case 4	N^* $T(N^*)$	39 232.27	40 210.97	41 197.40	42 187.99	41 223.59	41 203.50	41 190.12	41 180.57
Case 5	N^* $T(N^*)$	17 322.13	17 295.12	18 278.69	18 267.16	18 300.03	18 286.24	18 277.08	18 270.56

The optimal value of N, N^* and its minimum expected cost $T(N^*)$ are shown in Table 3 for $(\lambda,\mu,\beta,\theta,p,k)=(0.6,10,5,2,0.5,3)$ and for different values of (γ,α). It is noticed from Table 3 that (i) N* is insensitive and T(N*) decreases as γ increases from 0.3 to 0.6 and (ii) N* is insensitive and T(N*) increases as α increases from 0.05 to 0.20 for any case.

Table 3 The effect of (γ,α) on the optimal threshold value N* and the minimum expected cost for the five cost cases $(\lambda,\mu,\beta,\theta,p,k)=(0.6,10,5,2,0.5,3)$.

	(γ,α)	(.3, .1)	(.4, .1)	(.5, 0.)	(.6, .1)	(.4, .5)	(.4, .1)	(.4, .15)	(.4, .20)
Case 1	N^* $T(N^*)$	26 62.23	26 60.54	26 59.58	26 58.97	27 58.34	27 60.52	27 62.71	27 64.91
Case 2	N^* $T(N^*)$	41 105.62	41 102.61	41 100.87	41 99.73	42 98.45	42 102.61	42 106.77	42 110.95
Case 3	N^* $T(N^*)$	57 177.60	57 171.81	57 168.37	57 166.13	59 163.45	59 171.79	59 180.13	59 188.4
Case 4	N^* $T(N^*)$	40 209.56	40 203.53	40 200.03	40 197.75	41 195.17	41 203.50	41 211.86	41 220.21
Case 5	N^* $T(N^*)$	17 296.45	17 286.35	17 280.91	17 277.53	18 274.66	18 286.24	18 297.95	18 309.80

The optimum value of N, N* and its minimum expected cost T(N*) are shown in Table 4 for $(\lambda,\mu,\beta,\theta,\gamma,\alpha)=(0.6,10,5,2,0.4,0.1)$ and for different values of (p,k). From Table 4, we find that (i) N* and T(N*) increases as *p* decreases and (ii) N* increases and T(N*) decreases as *k* increases from 1 to 3 for any case.

Table 4 The effect of (p,k) on the optimal threshold value N* and the minimum expected cost for the five cost cases $(\lambda,\mu,\beta,\theta,\gamma,\alpha)=(0.6,10,5,2,0.4,0.1)$.

	(p,k)	1	0.5	0.25	(0.2, 1)	(0.2, 2)	(0.2, 3)
Case 1	N^* $T(N^*)$	21 31.53	27 60.52	32 96.95	15 80.91	22 67.02	27 60.52
Case 2	N^* $T(N^*)$	31 47.04	41 102.61	51 180.79	23 135.78	34 113.04	41 102.61
Case 3	N^* $T(N^*)$	44 64.60	58 171.79	64 340.43	33 219.67	47 186.75	58 171.79

Case 4	N^* $T(N^*)$	31 92.54	41 203.50	50 360.63	23 268.58	33 223.99	41 203.50
Case 5	N^* $T(N^*)$	14 136.74	18 286.24	21 441.59	10 410.49	14 326.54	18 286.24

Overall, we conclude that

- α, β, k and γ do not affect N^*
- θ rarely affects N^*
- λ affects N^* significantly
- C_h and C_s have much stronger effect on N^* than λ, θ, μ and α .

8. CONCLUSIONS

Two-phase N-policy $M^X/E_K/1$ queueing system with startup times and server breakdown was analyzed. Some of the system performance measures are derived. The cost function is formulated to determine the optimum value of N and hence minimum expected cost. Sensitivity analysis is performed for the optimum value of N, expected system length and minimum expected cost with various system parameters and cost elements for three specific batch size distributions viz. Deterministic, Geometric and positive Poisson. This queueing system is a generalization of the existing queueing systems of this type.

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