International Journal of Mathematical Archive-3(8), 2012, 2963-2970
IJMA Available online through www.ijma.info ISSN 2229 – 5046

UNIQUENESS OF MEROMORPHIC FUNCTIONS

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(Received on: 22-05-12; Revised & Accepted on: 04-08-12)

ABSTRACT

In this paper,we investigate the uniqueness of meromorphic functions concerning differential polynomials with weighted sharing method.Also study the uniqueness of meromorphic functions sharing a small function and a positive answer is given to the open problem possed by Dyavanal[11].

2000 Mathematics Subject Classification: *30D35.*

Keywords:*Uniqueness,Meromorphic,Shared Values.*

1. INTRODUCTION AND MAIN RESULTS

In this paper, meromorphic function means meromorphic in the complex plane.We adopt the standard notations in Nevanlinna theory of meromorphic functions as explained in [1,2]. Let E denote any set of positive real numbers of finite linear measure,not necessarily the same at each occurence.For a nonconstant meromorphic function f,we denote $T(r,f)$ the Nevanlinna characteristic of $f$and $S(r,f)$ any quantity satisfying

$S(r,f) = o\{T(r,f)\}$ $(r \to \infty, r \neq E)$.

Let f and g be two nonconstant meromorphic functions,and let a be a finite value. We say that f and g share the value a CM, provided that $f - a$ and $g - a$ have the same zeros with same multiplicities.Similiarly,we say that f and g share the value a IM,provided that $f - a$ and $g - a$ have the same zeros with ignoring multiplicities.

For convenience ,we give following notations and definitions .

For any constant a,we denote by $N_{k)}(r,\frac{1}{f-a})$ the counting function for zeros of $f(z) - a$ with multiplicity no more than k and $\overline{N}_{k)}(r,\frac{1}{f-a})$ the corresponding for which multiplicity is not counted.Let $N_{(k}(r,\frac{1}{f-a})$ be the counting function for zeros of $f(z) - a$ with multiplicity atleast k and $\overline{N}_{(k}(r,\frac{1}{f-a})$ the corresponding for which the multiplicity is not counted.

Set $N_k(r,\frac{1}{f-a}) = \overline{N}(r,\frac{1}{f-a}) + \overline{N}_{(2}(r,\frac{1}{f-a}) + \cdots + \overline{N}_{(k}(r,\frac{1}{f-a})$

We define,

$$\delta_k(a,f) = 1 - limsup_{r\to\infty}\frac{N_k(r,\frac{1}{f-a})}{T(r,f)}$$

$$\Theta(a,f) = 1 - limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}$$

Let l be non-negative integer or ∞. For any $a \in C \cup \infty$, we denote by $E_l(a,f)$ the set of all a-points of $f(z)$ where an a-points of multiplicity m is counted m times if $m \leq l$ and $l+1$ times if $m > l$.If $E_l(a,f) = E_l(a,g)$, we say that f and g share the value a with weight l. When $l = 0, f$ and g share 1 IM.[8]

In 2007, Bhoosnurmath and Dyavanal[3] proved the following theorem.

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Theorem A. Let f and g be two nonconstant meromorphic functions, and n, k be two positive integers with $n > 3k + 8$.If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM then either $f = tg$ for some n^{th} root of unity or $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$

Theorem B. Let f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{3}{n+1}$ and let n, k be two positive integers with $n \geq 3k + 13$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM,then $f(z) \equiv g(z)$

In 2008, A Banerjee[4] proved the following theorem.

Theorem C. Let f and g be two trancedental meromorphic functions and let n,k be two positive integers with $n > 9k + 14$.Suppose $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share a nonzero constant b IM,then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = b^2$ or $f = tg$ for some n^{th} root of unity.

In 2010, Pulak Sahoo[5] obtained the following result.

Theorem D. Let f and g be two trancedental meromorphic functions and let $n(\geq 1)$,$k(\geq 1)$ and $m(\geq 0)$ be three integers. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 IM.Then one of the following holds:

i) when $m = 0$,if $f(z) \neq \infty$, $g(z) \neq \infty$ and $n > 9k + 14$, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where $c, c_1$and c_2 are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.

ii) when $m = 1$, $n > 9k + 20$ and $\Theta(\infty, f) > \frac{2}{n}$, the either $[f^n(f-1)^m]^{(k)}[g^n(\mathrm{g}-1)^m]^{(k)} \equiv 1$ or $f = g$.

iii) when $m \geq 2$, $n > 9k + 4m + 16$ then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f = g$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$ where $R(x, y) = x^n(x-1)^m - y^n(y-1)^m$.

In 2011, Xiao Bin Zhang,JunFeng Xu[6] considered more general differential polynomial and obtained the following theorem:

Theorem E. Let f and g be two non constant meromorphic functions and $a(z)(\neq 0, \infty)$ be small function with respect to f. Let n, k and m be three positive integers with $n > 3k + m + 7$ and $P(\omega) = a_m \omega^m + a_{m-1} \omega^{m-1} + \cdots + a_0$ where $a_0 \neq 0, a_1, \cdots a_{m-1}, a_m \neq 0$ are complex constants.If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a CM, f and g share ∞ IM, then

i) $f(z) = tz$ for a constant t such that $t^d = 1$,

where $d = GCD(n+m, \cdots, n+m-i, \cdots, n), a_{m-i} \neq 0$, for some $i = 0,1, \cdots, m$

ii) f and g satisfy the algebraic equation $R(f, g) = 0$,

where $R(\omega_1, \omega_2) = \omega_1^n(a_m \omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m \omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$

iii)$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} = a^2$.

In 2009, using the notion of weighted sharing of values, Hong yan Xu and Ting Bin Cao[7] obtained following result.

Theorem F. Let f and g be two nonconstant entire functions and let m, n and k be three positive integers. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share

i) $(1,0)$ with $n \geq 5m + 5k + 8$

ii) $(1,1)$ with $n \geq \frac{9}{2}m + 4k + \frac{9}{2}$

iii) $(1,2)$ with $n \geq 3m + 3k + 5$

(1) when $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, then either $f = tg$, for a constant t such that $t^d = 1$ where $d = (n+m, \ldots, n+m-i, \ldots, n), a_{m-i} \neq 0$ for some $i = 0,1,2, \ldots, m$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where, $R(\omega_1, \omega_2) = \omega_1^n(a_m \omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m \omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$

(2) When $P(z) = 0$,then either $f = \frac{c_1}{\sqrt[n]{c_0}e^{cz}}$, $g = \frac{c_2}{\sqrt[n]{c_0}e^{-cz}}$. where c_1, c_2 and c are three constants satisfying

$(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for some constant t such that $t^n = 1$.

In this paper with the notion of weighted sharing of values,we investigate result for meromorphic function.

Theorem 1. Let f and g be two nonconstant transcendental meromorphic functions and let $n(\geq 1), k(\geq 1), l(\geq 0)$ be three integers.Let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0$ where $a_0 \neq 0, a_1, \cdots a_{m-1}, a_m \neq 0$ are complex constants.If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, l)$ and if

i) $l \geq 2$ and $n > 3k + 2m^* + m + 8$

ii) $l = 1$ and $n > 5k + 2m^* + m + 11$

iii) $l = 0$ and $n > 9k + 2m^* + 4m + 14$

then either
$f = tg$, for a constant t such that $t^d = 1$ where $d = (n + m, \ldots, n + m - i, \ldots, n), a_{m-i} \neq 0$ for some $i = 0,1,2, \ldots, m$ or f and g satisfy the algebraic equation $R(f, g) = 0$,

where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$

Theorem 2. Let f and g be two nonconstant entire functions and n, m and k be three positive integers.If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, l)$ and if

i) $l \geq 2$ and $n > 2k + m + 2m^* + 3$
ii) $l = 1$ and $n > 3k + 3m + 2m^* + 5$
iii) $l = 0$ and $n > 5k + 4m + 2m^* + 7$

then conclusion of Theorem 1 still holds.

In 2004, Lin and Yi[12] proved the following theorems.

Theorem G. Let f and g be two noncontant meromorphic funcitons,$n \geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $g = (n+2)(1-h^{n+1})/(n+1)(1-h^{n+2})$,

$f = (n+2)h(1-h^{n+1})/(n+1)(1-h^{n+2})$, where h is a nonconstant meromorphic function.

Theorem H. Let f and g be two noncontant meromorphic funcitons, $n \geq 13$ an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share the value 1 CM, then $f(z) = g(z)$.

In 2011, Renukadevi S Dyavanal [11] obtained following results.

Theorem I. Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n+1)s \geq 12$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$ for some (n+1) th root of unity d or $f(z) = c_2e^{-cz}$ and $g(z) = c_1e^{cz}$, where c, c_1 and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$

Theorem J. Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let n be an integer satisfying

$(n-2)s \geq 10$.

If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $g = (n+2)(1-h^{n+1})/(n+1)(1-h^{n+2})$,
$f = (n+2)h(1-h^{n+1})/(n+1)(1-h^{n+2})$,where h is a non-constant meromorphic function.

Theorem K. Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let n be an integer satisfying $(n-3)s \geq 10$. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share the value 1 CM, then $f \equiv g$.

At the end of this paper[11],she posed the question:Can the differential polynomials in theorems I, J and K be replaced by the differential polynomials of the form $[f^n]^{(k)}$ and $[f^n(f-1]^{(k)}$?

In this paper we consider more general differential polynomial of the form $[f^n P(f)]^{(k)}$, where $P(f)$ is as defined in Theorem 1, and give answer to open question(4.4) of [11]

Theorem 3. Let f and g be trancendental meromorphic functions,whose zeros and poles are of multiplicity atleast s. where s is a positive integer.$a(z)(\neq 0,\infty)$ be a small function with respect to f with finitely many zeros and poles. Let n, k and m be three positive integers satisfying $(n-m)s > 3k+7$.If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a CM and f and g share ∞ IM, then one of the following cases holds:

i)$f(z) = tg(z)$ for a constant t such that $t^d = 1$

where $d = (n+m,\dots,n+m-i,\dots,n)$,$a_{m-i} \neq 0$ for some $i = 0,1,\dots,m$

ii)f and g satisfy the algebraic equation $R(f,g) = 0$,

where, $R(\omega_1,\omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$

Remark:We set $P(z) = (z-1)^m$. With $a_m = 1, a_0 = -1$ and under condition (ii) of theorem 3, we have following important results.

i) When $m = 0$, $ns > 3k+7$ and if $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share a CM and ∞ IM then either $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$,where $c, c_1$and c_2 are constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.

ii) When $m = 1$, $(n-1)s > 3k+7$ and if $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share a CM and ∞ IM then $f \equiv g$.

iii) When $m \geq 2$, $(n-2)s > 3k+7$ and if $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share a CM and ∞ IM then f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(\omega_1,\omega_2) = \omega_1^n(\omega_1-1)^m - \omega_2^n(\omega_2-1)^m$.

Remarks (i),(ii) and (iii) give answers to open problem (4.4) of [11].

2. LEMMAS

In order to prove our results, we need the following lemmas.

Lemma1 [1]. Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$, where $a_n(\neq 0)$, $a_{n-1},\dots,a_0$ are constants. If $f(z)$ is a meromorphic function, then

$$T(r,P(f)) = nT(r,f) + S(r,f).$$

Lemma 2[12]. Let $f(z)$ a nonconstant meromorphic and p,k be two positive integer. Then

$$N_p(r,\frac{1}{f^{(k)}}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f)$$
$$N_p(r,\frac{1}{f^{(k)}}) \leq k\overline{N}(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f)$$

Lemma 3 [2]. Let $f(z)$ be nonconstant meromorphic functions and k be a positive integer.Suppose that $f^{(k)} \neq 0$, then

$$N(r,\frac{1}{f^{(k)}}) \leq N(r,\frac{1}{f}) + k\overline{N}(r,f) + S(r,f)$$

Lemma 4[6]. Let $f(z)$ and $g(z)$ two be nonconstant meromorphic function and n,k be two positive integers and a be a finite nonzero constant.If $f(z)$ and $g(z)$ share a CM and ∞IM,then one of the following cases holds:

i)$T(r,f) \leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + 3\overline{N}(r,f) + S(r,f) + S(r,g)$the same inequality holds for T(r,g);
ii)$fg = a^2$,
iii)$f \equiv g$.

Lemma 5[13]. Let $f(z)$ and $g(z)$ two be nonconstant meromorphic functions, $k(\geq 1), l(\geq 0)$ be two integers.Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1,l)$. If one of the following conditions holds,

i) $l \geq 2$ and $\Delta_1 = 2\Theta(\infty,f) + (k+2)\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) > (k+7)$

ii) $l = 1$ and $\Delta_2 = (k+3)\Theta(\infty, f) + (k+2)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 2k + 9$
iii) $l = 0$ and $\Delta_3 = (2k+4)\Theta(\infty, f) + (2k+3)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta_{k+1}(0, g) > 4k + 13$ then either $f^{(k)}g^{(k)} \equiv 1$ or $f(z) = g(z)$.

Taking $N(r, f) = N(r, g) = 0$ and proceeding as in lemma 6[12], we get following lemma.

Lemma 6. Let $f(z)$ and $g(z)$ two be nonconstant entire functions, $k(\geq 1), l(\geq 0)$ be two integers.Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1, l)$. If one of the following conditions holds,
i)$l \geq 2$ and $\Delta_4 = \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 3$
ii)$l = 1$ and $\Delta_5 = \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 4$
iii)$l = 0$ and $\Delta_6 = \Theta(0, f) + \Theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta_{k+1}(0, g) > 6$ then either $f^{(k)}g^{(k)} \equiv 1$ or $f(z) = g(z)$.

Lemma 7[6]. let f and g be two nonconstant meromorphic functions, let n and k be two integers with $n > k + 2$, let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0$ where $a_0 \neq 0, a_1, \cdots a_{m-1}, a_m \neq 0$ are constants, and let $a(z)(\neq 0, \infty)$ be small function with respect to f with finitely many zeros and poles.

If $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} = a^2$ and f and g share ∞ IM,then $P(z)$ is reduced to a nonzero monomial,namely,$P(z) = a_i z^i \neq 0$ for some i = 0,1,... , m.

3. PROOF OF THEOREMS

Proof of theorem 1.

Let $F = f^n P(f)$ and $G = g^n P(g)$

Then we have,

$$\begin{aligned}\Theta(0, f) &= 1 - \overline{lim}_{r\to\infty} \frac{\overline{N}(r, 1/F)}{T(r, F)} \\ &= 1 - \overline{lim} r \to \infty \frac{\overline{N}(r, 1/f^n) + \overline{N}(r, 1/P(f))}{T(r, F)} \\ &\geq 1 - \frac{(1+m^*)T(r, f)}{(n+m)T(r, f)},\end{aligned}$$

Therefore, $\Theta(0, f) \geq \frac{n+m-1-m^*}{n+m}$

where $m^* = 0$ if m =0 and $m^* = 1$ if $m \geq 1$

Similarly, $\Theta(0, G) \geq \frac{n+m-1-m^*}{n+m}$

Next we have ,

$$\begin{aligned}\delta_{k+1}(0, f) &= 1 - \overline{lim}_{r\to\infty} \frac{N_{k+1}(r, 1/F)}{T(r, F)} \\ &\geq 1 - \overline{lim}_{r\to\infty} \frac{(k+1)\overline{N}(r, 1/f) + N_{k+1}(r, 1/P(f))}{(n+m)T(r, f)} \\ &\geq 1 - \overline{lim}_{r\to\infty} \frac{(k+1)\overline{N}\left(r, \frac{1}{f}\right) + mT(r, f)}{(n+m)T(r, f)} \\ &\geq 1 - \frac{m+k+1}{n+m}\end{aligned}$$

Therefore, $\delta_{k+1}(0, f) \geq \frac{n-k-1}{n+m}$

Similarly, $\delta_{k+1}(0, G) \geq \frac{n-k-1}{n+m}$

We have,
$$\begin{aligned}\Theta(\infty, f) &= 1 - \overline{lim}_{r\to\infty} \frac{\overline{N}(r, F)}{T(r, F)} \\ &= 1 - \overline{lim} r \to \infty \frac{\overline{N}(r, f)}{T(r, F)} \\ &\geq 1 - \frac{T(r, f)}{(n+m)T(r, f)},\end{aligned}$$

Therefore, $\Theta(\infty, f) \geq \frac{n+m-1}{n+m}$

Since $F^{(k)}$ and $G^{(k)}$ share $(1,l)$ we consider following three cases.

Case 1: Let $l \geq 2$,

$$\begin{aligned}\Delta_1 &= (k+2)\Theta(\infty,G) + 2\Theta(\infty,F) + \Theta(0,F) + \Theta(0,G) + \delta_{k+1}(0,F) + \delta_{k+1}(0,G)\\ &\geq (k+4)\left(\frac{n+m-1}{n+m}\right) + 2\left(\frac{n+m-1-m^*}{n+m}\right) + 2\left(\frac{n-k-1}{n+m}\right)\\ &= (k+4)\left(1-\frac{1}{n+m}\right) + 2\left(1-\frac{1+m^*}{n+m}\right) + 2\left(\frac{n-k-1}{n+m}\right)\\ &= (k+6) - \left(\frac{k+4}{n+m} + \frac{2+2m^*}{n+m} - \frac{2(n-k-1)}{n+m}\right)\\ &= (k+8) - \left(\frac{3k+2m^*+2m+8}{n+m}\right)\end{aligned}$$

from(i) of lemma(5), we have $n+m \leq 3k+2m^*+2m+8$ i.e $n \leq 3k+2m^*+m+8$

which contradicts our hypothesis that $n > 3k+2m^*+m+8$.

By lemma (5),we have either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$

Case 2: Let $l = 1$

$$\begin{aligned}\Delta_2 &= (k+2)\Theta(\infty,G) + (k+3)\Theta(\infty,F) + \Theta(0,F) + \Theta(0,G) + 2\delta_{k+1}(0,F) + \delta_{k+1}(0,G)\\ &\geq (2k+5)\left(\frac{n+m-1}{n+m}\right) + 2\left(\frac{n+m-1-m^*}{n+m}\right) + 3\left(\frac{n-k-1}{n+m}\right)\\ &= (2k+5)\left(1-\frac{1}{n+m}\right) + 2\left(1-\frac{1+m^*}{n+m}\right) + 3\left(\frac{n-k-1}{n+m}\right)\\ &= (2k+7) - \left(\frac{2k+5}{n+m} + \frac{2+2m^*}{n+m} - \frac{3(n-k-1)}{n+m}\right)\\ &= (2k+10) - \left(\frac{5k+2m^*+3m+10}{n+m}\right)\end{aligned}$$

From (ii) of lemma(5), we have $n \leq 5k+2m^*+2m+10$

which contradicts our hypothesis that $n > 5k+2m^*+m+10$.

By lemma (5),either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Case 3: let $l = 0$

$$\begin{aligned}\Delta_3 &= (2k+4)\Theta(\infty,F) + (2k+3)\Theta(\infty,G) + \Theta(0,F) + \Theta(0,G) + 3\delta_{k+1}(0,F) + 2\delta_{k+1}(0,G)\\ &\geq (4k+7)\left(\frac{n+m-1}{n+m}\right) + 2\left(\frac{n+m-1-m^*}{n+m}\right) + 5\left(\frac{n-k-1}{n+m}\right)\\ &= (4k+7)\left(1-\frac{1}{n+m}\right) + 2\left(1-\frac{1+m^*}{n+m}\right) + 5\left(\frac{n-k-1}{n+m}\right)\\ &= (4k+9) - \left(\frac{4k+7}{n+m} + \frac{2+2m^*}{n+m} - \frac{5(n-k-1)}{n+m}\right)\\ &= (4k+14) - \left(\frac{9k+2m^*+5m+14}{n+m}\right)\end{aligned}$$

From (iii) of lemma(5), we have $n \leq 9k+2m^*+4m+14$

which contradicts our hypothesis that $n > 5k+2m^*+m+10$.

By lemma (5), either $F^{(k)}G^{(k)} \equiv 1$, or $F \equiv G$.

Suppose $F^{(k)}G^{(k)} \equiv 1$ then by lemma (7), $P(z)$ as defined in Theorem 1 reduces to a nonzero monomial. That is

$P(z) = a_i z^i \neq 0$ for some $i = 01,2,\ldots,m$.

By hypothesis of theorem (1), we arrive at a contradiction.

Hence we deduce that $F(z) \equiv G(z)$, that is

$f^n(a_m f^m + a_{m-1}f^{m-1} \ldots + a_0) = g^n(a_m g^m + a_{m-1}g^{m-1} \ldots + a_0)$

Let $h = f/g$.If h is a constant then substituting $f = gh$, we deduce,

$a_m g^{n+m}(h^{n+m}-1) + a_{m-1}g^{n+m-1}(h^{n+m-1}-1) + \cdots + a_0 g^n(h^n-1) = 0$

which implies that $h^d = 1$ where $d = (n+m, \dots, n+m-i, \dots, n)$,$a_{m-i} \neq 0$ for some $i = 0,1,\dots,m$

Thus $f(z) = tg(z)$ for a constant t such that $t^d = 1$,

where $d = (n+m, \dots, n+m-i, \dots, n)$,$a_{m-i} \neq 0$, for some $i = 0,1,\dots,m$.

If h is not a constant then f and g satisfy the algebraic equation $R(f,g) = 0$,

where $R(\omega_1,\omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \dots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \dots + a_0)$

This proves the theorem.

Proof of theorem 2.
Since f and g are entire functions $N(r,f) = N(r,g) = 0$. Proceeding as in theorem 1 and using lemma (5) ,we easily prove theorem 2.

Proof of theorem 3.

Let $F = [f^nP(f)]^{(k)}$, $G = [g^nP(g)]^{(k)}$, $F_1 = F/a$, $G_1 = G/a$,
$F^* = f^nP(f)$, $G^* = g^nP(g)$

then by hypothesis F_1 and G_1 share 1 CM.

By case(i) of lemma (4),we have

$$T(r,F) \leq N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G}) + 3\overline{N}(r,F) + S(r,F) + S(r,G) \tag{1}$$

By lemma(2),with $p = 2$,we obtain,

$$T(r,F^*) \leq T(r,F) - N_2(r,\tfrac{1}{F}) + N_{k+2}(r,\tfrac{1}{F^*}) + S(r,F) \tag{2}$$

$$N_2(r,\tfrac{1}{G}) \leq N_{k+2}(r,\tfrac{1}{G^*}) + k\overline{N}(r,G) + S(r,G) \tag{3}$$

By (1) and (2),we have

$$T(r,F^*) \leq N_2\left(r,\frac{1}{G}\right) + 3\overline{N}(r,F) + N_{k+2}\left(r,\frac{1}{F^*}\right) + S(r,F) + S(r,G)$$

using (3), we get

$$T(r,F^*) \leq N_{k+2}(r,\frac{1}{G^*}) + k\overline{N}(r,G) + 3\overline{N}(r,F) + N_{k+2}(r,\frac{1}{F^*}) + S(r,F) + S(r,G)$$
$$\leq (k+2)\overline{N}\left(r,\tfrac{1}{g}\right) + N\left(r,\tfrac{1}{P(G)}\right) + k\overline{N}(r,G) + 3\overline{N}(r,F) + (k+2)\overline{N}\left(r,\tfrac{1}{f}\right) + N(r,\tfrac{1}{P(f)}) + S(r,F) + S(r,G)$$

By our assumption,zeros and poles are of multiplicities at least s, that is,
$\overline{N}(r,g) \leq \frac{1}{s}N(r,g) \leq \frac{1}{s}T(r,g)$ and we deduce the above inequality as,

$$T(r,F^*) \leq \left(\frac{k+2}{s}\right)T(r,g) + mT(r,g) + \frac{k}{s}T(r,g) + \frac{3}{s}T(r,f) + (\frac{k+2}{s})T(r,f) + mT(r,f) + S(r,F) + S(r,G)$$
$$\leq \left(\tfrac{k+2}{s} + \tfrac{3}{s} + m\right)T(r,f) + \left(\tfrac{k+2}{s} + \tfrac{k}{s} + m\right)T(r,g) + S(r,F) + S(r,G)$$

$$(n+m)T(r,f) \leq \left(\frac{ms+k+5}{s}\right)T(r,f) + \left(\frac{2k+ms+2}{s}\right)T(r,g) + S(r,F) + S(r,G)$$

$$(ns-k-5)T(r,f) \leq (2k+ms+2)T(r,g) + S(r,F) + S(r,G)$$

Similarly,

$$(ns-k-5)T(r,g) \leq (2k+ms+2)T(r,g) + S(r,F) + S(r,G)$$

$$(ns-k-5)(T(r,f) + T(r,g)) \leq (2k+ms+2)(T(r,f) + T(r,g) + S(r,f) + S(r,g)$$

$(ns - ms - 3k - 7)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)$

which contradicts $(n - m)s > 3k + 7$

Therefore by Lemma(4), either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Proceeding as in proof of theorem 1 we obtain theorem 3.

6. ACKNOWLEDGMENT

This research work is supported by Department of Science and Technology Government of India ,Ministry of Science and Technology,Technology Bhavan,New Delhi under the sanction letter No $(SR/S4/MS{:}\,520/08)$

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Source of support: Department of Science and Technology Government of India, Ministry of Science and Technology,Technology Bhavan, New Delhi, India, Conflict of interest: None Declared