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# THE EASY WHITNEY EMBEDDING THEOREM IN THE COMPLEX CASE

**Hamdi M. Genedi**

*Mathematics and Statistics Department, Faculty of Science, Taif University, Kingdom of Saudi Arabia*
*E-mail: hmgenedy1954@hotmail.com*

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*ABSTRACT*

*Whitney proved that if $M$ is compact n-dimensional manifold with analytic structure, then there is an analytic structure embedding of $M$ in $\mathbb{R}^{2n+1}$. In this paper we prove this theorem in the complex case.*

***Keywords:*** *Embeddings, Immersions, complex manifolds and analytic structure.*

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## 1. INTRODUCTION:

In [4], Whitney proved that if $M$ is compact Hausdorff $C^r$ n-dimensional manifold, $2 \le r \le \infty$, then there is a $C^r$ embedding of $M$ in $\mathbb{R}^{2n+1}$. The aim of this paper is to prove the theorem but in complex case, i.e., if $M$ is compact complex n-dimensional manifold with analytic structure, then there is an analytic structure embedding of $M$ in $\mathbb{C}^{n+2}$. The definitions and fundamental concepts which will be required throughout the paper may be found in [1, 2, 3].

## 2. MAIN THEOREM:

We need the following lemmas:

**Lemma: 1** Let M be an analytic compact complex n-dimensional manifold. Then there exists an analytic embedding of M into $\mathbb{C}^n$.

**Proof:** $D^n(r) = \{z \in \mathbb{C}^n : |z| \le r\}$, this is closed disk of radius r and centre o in $\mathbb{C}^n$. Since M is compact it satisfies the finite intersection property and so one can easily find an atlas $\{\varphi_i, U_i\}_{i=1}^{n}$ having the following properties:

**(a)** for every $\varphi_i(U_i) \supset D^n(2)$, where $D^n(2)$ is the interior of the unit complex sphere $S^n$.
**(b)** $M = \bigcup \mathrm{Int}\, \varphi_i^{-1}(\Delta^n)$, where $\Delta^n$ denotes a unit closed disk in $\mathbb{C}^n$, i.e. $\Delta^n = D^n(1)$.

Let $\lambda : \mathbb{C}^n \to S^2$ be an analytic map such that $\lambda \mid \Delta^n = |z|$, $(\lambda \mid \mathbb{C}^n) \setminus D^n(2) = 0$. Define an $C^\omega$- map $\lambda_i : M \to S^2$ such that:

$$\lambda_i = \begin{cases} \lambda\ \varphi_i & on\ U_i \\ 0 & on\ U_i^c \end{cases}$$

It follows that the sets $B_i = \lambda_i^{-1}(|z|) \subset U_i$ cover $M$. Define maps $f_i : M \to \mathbb{C}^n$ such that:

$$f_i = \begin{cases} \lambda_i(z)\, \varphi_i(z) & if \quad z \epsilon U_i \\ 0 & if \quad z \epsilon U_i^c \end{cases}$$

Put $g_i = (f_i, \lambda_i) \to \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1}$ and $g = (g_1, \ldots, g_m) : \mathrm{M} \to \underbrace{\mathbb{C}^{n+1} \times \ldots \times \mathbb{C}^{n+1}}_{m-times} = \mathbb{C}^{m(n+1)}$. Clearly $g$ is $C^\omega$ map. If $z_1 \in B_i$ then $g_i$, and hence $g$, is immersive at $z_1$, so $g$ is an immersion. To see that $g$ is injective, suppose $z_1 \neq z_2$ with $z_2 \in B_i$. If $z_1 \in B_i$ then $g(z_1) = g(z_2)$ since $f_i \mid B_i = \varphi_i \mid B_i$. If $z_1 \notin B_i$ then $\lambda_i(z_2) = 1 \neq \lambda_i(z_1)$, so $g(z_1) \neq g(z_2)$. Therefore $g$ is an injective $C^\omega$- immersion. Since $M$ is compact, then $g$ is an embedding.

**Lemma: 2** Let $M^m$ and $N^n$ be complex manifolds with $n > m$. If $f : M^m \to N^n$ is a $C^\omega$- map, then $f(M^m)$ is nowhere dense.

**Proof:** It suffices to show that $f(M^m)$ has measure zero this follows from:
If $g : U \to \mathbb{C}^n$ which is analytic and with $m < n$ and if $U \subset \mathbb{C}^m$ is open, then $g(U) \subset \mathbb{C}^n$ has measure zero. To prove

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***Corresponding author: Hamdi M. Genedi, E-mail: hmgenedy1954@hotmail.com**

this assertion, write $g$ as a composition of $C^{\omega}$- maps $U = U \times \{0\} \subset U \times \mathbb{C}^{n-m} \xrightarrow{\pi} U \xrightarrow{g} \mathbb{C}^n$. $U \times \{0\}$ has $n$-measure zero in $U \times \mathbb{C}^{n-m} \subset \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^n$. Now every point of $V$, where $V \subset U$, belongs to an open ball $B \subset U$ such that the norm $\|Dg(z)\|$ is uniformly bounded on $B$, say $\beta > 0$, where $\beta$ is a real number. Then $|f(z) - f(z\ )| \leq \beta|z - z|$ for all $z, z \in B$. It follows that if $K \subset B$ is an $n$-cube of edge ƛ, then $f(K)$ is contained in an n-cube $K$ of edge less than ƛ$\sqrt{n\beta}$=ƛ$L$. Therefore $\mu(K\ ) < L^n\mu(K)$. Now write $V = \bigcup_{j=1}^{\infty} V_j$ where each $V_j$ is a compact subset of a ball $B$ as above. For each $\varepsilon > 0$, $V_j = \bigcup_i K_i$, where each $K_i$ is an $n$-cube in $\mathbb{C}^n$ and $\sum \mu(K_i) < \varepsilon$. It follows that $f(V_j) \subset \bigcup_i K_i$ where the sum of the measure of the $n$-cube $K_i$ is less than $L^n\varepsilon$. Hence $f(V_j)$ has measure zero, and so $V$ has measure. Then $f(M)$ has measure zero by prolongation.

The inverse function theorem 3 ([5]).Suppose

**(a)** $W$ is an open subset of a banach space $X$,
**(b)** $f: W \to X$ is continuously differentiable,
**(c)** for every $z \in W$ and $(Df)_z$ is an invertible member of the collection of all bounded linear mappings of $X$ into $X$.

Every point $a \in W$ has then a neighbourhood $U$ such that

**(i)** $f$ is one-to-one in $U$,
**(ii)** $f(U)$= $V$ is an open subset of $X$, and
**(iii)** $f^{-1}: V \to U$ is uniformly continuous.

**Lemma: 4** Let $N^n$ be a $C^{\omega}$-manifold. A subset $A \subset N^n$ is a $C^{\omega}$- submanifold if and only if $A$ is the image of a $C^{\omega}$-embedding .

**Proof:** Suppose $A$ is a $C^{\omega}$- manifold. Then $A$ has a natural analytic structure derived from a covering by submanifold charts. For this analytic structure, the inclusion of $A$ in $N^n$ is $C^{\omega}$- embedding. Conversely, suppose $f: M^m \to N^n$ is a $C^{\omega}$- embedding, $f(M^m) = A$. The property of being a $C^{\omega}$- submanifold has local character, That is true if $A \subset M^m$ if and only if it is true if $A_i \subset N_i$ where $\{A_i\}$ is an open cover of $A$ and each $N_i$ is an open subset of $M^m$ containing $A_i$. It is also invariant under $C^{\omega}$ -holeomorphic map, that is, $A \subset N$ is a $C^{\omega}$ - submanifold if and only if $g(A) \subset N$ is a $C^{\omega}$- submanifold where $g: N \to N$ is a $C^{\omega}$-holeomorphic map (or even a $C^{\omega}$-embedding). To exploit local character and invariance under holeomorphic map, let $\Psi = \{_i: N_i \to \mathbb{C}^n\}_{i\in}$ be a family of charts on $N^n$ which covers $A$. Then find an atlas $\Phi = \{\varphi_i: M_i \to \mathbb{C}^m\}_{i\in}$ for $M^m$ such that $f(M_i) \subset N_i$ . Since $f$ is an embedding, $\Phi$ and $\Psi$ can be chosen so that $f(M_i) = A \cap N_i$ . By invariance it is enough to show that ${}_i f(M_i) \subset \mathbb{C}^n$ is a $C^{\omega}$ - submanifold. Put $U_i = \varphi_i(M_i) \subset \mathbb{C}^m$,

$f_i = {}_i \circ f \circ \varphi_i^{-1}: U_i \to \mathbb{C}^n$. Then $f_i$ is a $C^{\omega}$- embedding and $g_i(U_i) = {}_i \circ f(M_i)$. Thus we have reduced the lemma to the special case where $N = \mathbb{C}^n$, $M$ is an open set $U \subset \mathbb{C}^m$, and $f: U \to \mathbb{C}^n$ is a $C^{\omega}$- embedding. In this case a corollary of the inverse function theorem implies that there is a $C^{\omega}$- submanifold chart for $(\mathbb{C}^n, f(U))$ at each point of $f(U)$.

**Lemma: 5** Let $f: M \to N$ be a $C^{\omega}$-map. If $z \in f(M)$ is an analytic value i.e. it satisfies the Cauchy-Riemann equations. Then $f^{-1}(z)$ is a $C^{\omega}$- submanifold of $M$.

**Proof:** By using local character and invariance, as in the proof of lemma 4, we reduce the lemma to the case where $M$ is an open set in $\mathbb{C}^m$ and $N = \mathbb{C}^n$. Again the lemma follows from the inverse function theorem.

**Theorem: (Main)** Let $M$ be a compact complex $n$- dimensional manifold with analytic structure. Then there is an analytic structure embedding of $M$ in $\mathbb{C}^{n+2}$.

**Proof:** By lemma 1, $M$ embeds in some $\mathbb{C}^q$. If $q = n + 1$ there is nothing to prove; hence we assume the case that $q > n + 1$. We may replace $M$ by its image under an embedding. Therefore we assume that $M$ is a $C^{\omega}$- submanifold of $\mathbb{C}^q$. It is sufficient to prove that such an $M$ embeds in $\mathbb{C}^{q-1}$, for repetition of the argument will eventually embed $M$ in $\mathbb{C}^{n+1}$.

Suppose then that $M \subset \mathbb{C}^q$, q> $n + 1$. Identify $\mathbb{C}^{q-1}$ with $\{z \in \mathbb{C}^q : z_q = 0\}$. If $\underline{v} \in \mathbb{C}^q - \mathbb{C}^{q-1}$ denote by $f_{\underline{v}}: \mathbb{C}^q \to \mathbb{C}^{q-1}$ the projection parallel to $\underline{v}$ . We seek a vector $\underline{v}$ such that $f_{\underline{v}} \mid M : M \to \mathbb{C}^{q-1}$ is a$C^{\omega}$- embedding. We limit our search to unit vectors.

For $f_{\underline{v}} \mid M$ to be injective means that $\underline{v}$ is not parallel to any secant of $M$. That is, if $z_1, z_2$ are any two distinct points of $M$, then

$$\underline{v} \neq \frac{z_1 - z_2}{|z_1 - z_2|} \cdots \tag{1}$$

More subtle is the requirement that $f_{\underline{v}} \mid M$ be an immersion. The kernel of the analytic map $f_{\underline{v}}$ is obviously the circle $|z| = 1$ through $\underline{v}$. Therefore a tangent vector $z \in M_z$ is in the kernel of $T_z \circ f_{\underline{v}}$ only if $z$ is parallel to $\underline{v}$. We can guarantee that $f_{\underline{v}} \mid M$ is an immersion by requiring, for all nonzero $z \in T(M)$:

$$\underline{v} \neq \frac{z}{|z|} \cdots \tag{2}$$

Here $z$ is identified with a vector in $\mathbb{C}^q$; thus $|z|$ makes sense. Condition (1) is analyzed by means of the map

$$\rho : M \times M - \Delta \to S^{q-1},$$
$$\rho\,(z_1, z_2) = \frac{z_1 - z_2}{|z_1 - z_2|},$$

where $\Delta$ is the diagonal, i.e.,

$$\Delta = \{\, (a, b) \in M \times M : a = b \,\}.$$

Now $\underline{v}$ satisfies (1) if and only if $\underline{v}$ is not in the image of $\rho$; we consider $M \times M - \Delta$ as an open submanifold of $M \times M$; the map $\rho$ is then $C^{\omega}$. Note that

$$\dim\,(M \times M - \Delta) = 2n < \dim S^{q-1}.$$

The existence of a $\underline{v}$ satisfying (1) follows from lemma 2. In the case at hand, put $M \times M - \Delta$ and $S^{q-1}$ instead of $M^m$ and $N^n$, respectively, in lemma 2. We know that every nonvoid open subset of $S^{q-1}$ contains a point $\underline{v}$ which is not in the image of $\rho$. To analyze condition (2) we note that it holds for all $z \in T(M)$ provided it holds whenever $|z| = 1$. Let

This is the unit tangent bundle of $M$. It is $C^{\omega}$- submanifold of $T(M)$. To see this, observe that $T_1(M) = \underline{v}^{-1}(|z|)$ where $\underline{v} : T(M) \to \mathbb{C}$, $\underline{v}$ is defined by $\underline{v}(z) = |z|^2$. Since $\underline{v}$ is the restriction to $T(M)$ of the $C^{\omega}$- map $T(\mathbb{C}^q) \to \mathbb{C}$, $z \to |z|^2$, it is $C^{\omega}$. It is clear that (1, 0) is a regular value for $\underline{v}$; for if $\underline{v}(z) = 1$, then

$$\frac{d}{dt}\ \underline{v}(tz)|_{t=1} \neq 0.$$

Hence $\underline{v}^{-1}(|z|)$ is a $C^{\omega}$- submanifold by lemma 4. It is easy to see that it is compact because $M$ is compact. Define a $C^{\omega}$- map $\tau : T_1(M) \to S^{q-1}$ as follows. Identify $T(M)$ with a subset of $M \times \mathbb{C}^q$; then $T_1(M)$ is a subset of $M \times S^{q-1}$. Define $\tau$ to be the restriction to $T_1(M)$ of the projection onto $S^{q-1}$. Geometrically $\tau$ is just parallel translation of unit vectors based at points of $M$ to unit vectors based at 0. Clearly $\tau$ is $C^{\omega}$. Noting that $\dim T_1(M) = n - 1 < \dim S^{q-1}$, we apply lemma (2) to conclude that the image of $\tau$ is nowhere dense. Since $T_1(M)$ is compact, it follows that the complement $W$ of the image of $\tau$ is a dense open set in $S^{q-1}$. Therefore $W$ meets $S^q \cap (\mathbb{C}^q - \mathbb{C}^{q-1})$ in a nonempty open set $W_0$. As we have seen previously, $W_0$ contains a vector $\underline{v}$ which is not in the image of $\rho$. This vector $\underline{v}$ has the property that $f_{\underline{v}} \mid M : M \to \mathbb{C}^q$ is an injective immersion. Since $M$ is compact and Hausdorff, $f_{\underline{v}} \mid M$ is also an embedding.

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