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COMMUTANT OF THE DIRECT SUM OF MULTIPLICATION OPERATORS

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ABSTRACT

For $i=1,\ldots,n$, suppose that B_i is Banach space of analytic functions on a bounded domain G_i in the complex plane, and $G_i \cap G_j = \phi$ for $i \neq j$. Let M_i denote the operator of multiplication by z on B_i. It is shown that the commutatnt and the double commutatnt of $M_1 \oplus \cdots \oplus M_n$ are equal; furthermore, the commutant of $M_1 \oplus \cdots \oplus M_n$ split. That is, $\{M_1 \oplus \cdots \oplus M_n\}' = \{M_1\}' \oplus \cdots \oplus \{M_n\}'$. Also, we prove that the direct sum of an upper or lower triangular operator on $\mathrm{B}_1 \oplus \cdots \oplus \mathrm{B}_1$ and another one on $\mathrm{B}_2 \oplus \cdots \oplus \mathrm{B}_2$ split.

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1. INTRODUCTION

Let G be bounded domain in the complex plane $\mathbb{C}$. Suppose that B is a reflexive Banach space of analytic functions on G. Denote the operator of multiplication by z on B by M_z. The operator M_z and many properties of this operator have been studied in literature (see, for example, [1], [3-12]).

A complex-valued function ϕ on G is called a multiplier of B if $\phi\mathrm{B} \subseteq \mathrm{B}$. The set of all multipliers will be denoted by $M(B)$. It is known that $M(\mathrm{B}) \subseteq \mathrm{B} \cap H^{\infty}(G)$, whenever $H^{\infty}(G)$ is the space of bounded analytic functions on G with the supremum norm. ([2, Proposition 3]). Each multiplier ϕ of B determines a multiplication operator M_ϕ defined by $M_\phi f = \phi f$, for all $f \in \mathrm{B}$. For every scalar λ, let us denote by e_λ the functional of evaluation at λ on B, defined by $e_\lambda f := \langle f, e_\lambda \rangle = f(\lambda)$. It is well known that $M_\phi^* e_\lambda = \phi(\lambda) e_\lambda$.

2. MAIN RESULTS

If T is a bounded linear operator on a Banach space B, the commutant of T, denoted by $\{T\}'$, consists of all bounded linear operators on B which commute with T. That is, $\{T\}' = \{S \in L(\mathrm{B}) : ST = TS\}$. Let $\{T\}''$ denote the double commutant of T; $i.e.$, $\{\{T\}'\}'$. In [9] and [10], the commutant of the direct sum of some operators on certain Banach spaces of functions are characterized. For $i=1,2,\cdots,n$, suppose that B_i is a Banach space of analytic functions on the bounded domains G_i in the complex plane $\mathbb{C}$. Let M_i be the operator of multiplication by z on B_i defined by $(M_i f)(z) = zf(z)$, for every $f \in \mathrm{B}_i$, such that $\sigma(M_i) = \overline{G_i}$, and dim $\ker(M_i - \lambda)^* = 1$ for every $\lambda \in G_i$.

It follows from [12] that $\{M_i\}' = \{M_\phi : \phi \in M(\mathrm{B}_i)\}$, for $i=1,2,\cdots,n$. The matrix of every operator S acting on the Banach space $\mathrm{B}_1 \oplus \mathrm{B}_2 \oplus \cdots \oplus \mathrm{B}_n$ can be written as $S = (S_{ij})_{1 \le i,j \le n}$, where the operator $S_{ij} : \mathrm{B}_j \to \mathrm{B}_i$, $i,j = 1,2,\cdots,n$ is defined by $S_{ij} = P_i S|_{B_j}$ in which P_i is the projection from $B_1 \oplus \cdots \oplus B_n$ onto B_i.

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An operator $X \in L(B_2, B_1)$ is said to intertwines the operators $T_1 \in L(B_1)$ and $T_2 \in L(B_2)$ if and only if $T_1 X = X T_2$.

Theorem 1. If $G_i \cap G_j = \phi$, for all $i, j = 1, \cdots, n$ with $i \neq j$, then the algebra $\{M_1 \oplus \cdots \oplus M_n\}'$ splits; that is,

$$\{M_1 \oplus \cdots \oplus M_n\}' = \{M_1\}' \oplus \cdots \oplus \{M_n\}'.$$

Furthermore,

$$\begin{aligned}\{M_1 \oplus \cdots \oplus M_n\}'' &= \{M_1 \oplus \cdots \oplus M_n\}' \\ &= \{M_{\phi_1} \oplus \cdots \oplus M_{\phi_n} : \phi_i \in M(B_i), i = 1, \cdots, n\}.\end{aligned}$$

Proof: Suppose that $S = (S_{ij})_{1 \le i, j \le n} \in \{M_1 \oplus \cdots \oplus M_n\}'$. The matrix of the operator $M_1 \oplus \cdots \oplus M_n$ is a diagonal matrix $M = (M_{ij})_{1 \le i, j \le n}$ where $M_{ii} = M_i$, and $M_{ij} = 0$ for $i \neq j$. Then the operator S_{ij} intertwines M_i and M_j; $i.e., M_i S_{ij} = S_{ij} M_j$. Consequently, $S_{ii} \in \{M_i\}' = \{M_\phi : \phi \in M(B_i)\}$. Also, S_{ij}^* intertwines M_i^* and M_j^*. So, if $\lambda \in G_i$ then $M_j^* S_{ij}^* e_\lambda = \lambda S_{ij}^* e_\lambda$. If $i \neq j$ and $S_{ij}^* e_\lambda \neq 0$, then $\lambda \in \sigma(M_j^*) = \sigma(M_j) = \overline{G_j}$, which is a contradiction, because $G_i \cap G_j = \phi$. Thus, if $i \neq j$, then for all $\lambda \in G_i$, we see that $S_{ij}^* e_\lambda = 0$.

Now, since the linear span of $\{e_\lambda : \lambda \in G_i\}$ is dense in B_i^*, we conclude that $S_{ij}^* = 0$, and so $S_{ij} = 0$, for $i \neq j$. It follows that

$$\begin{aligned}\{M_1 \oplus \cdots \oplus M_n\}' &= \{\hat{M}\ \phi_1 \oplus \cdots \oplus M_{\phi_n} : \phi_i \in M(B_i)\} \\ &= \{M_1\}' \oplus \cdots \oplus \{M_n\}'.\end{aligned}$$

To show that $\{M_1 \oplus \cdots \oplus M_n\}'' = \{M_1 \oplus \cdots \oplus \neq M_n\}'$, first note that $\{M_i\}'' \subseteq \{M_i\}'$, for $i = 1, \cdots, n$, and $\{M_1 \oplus \cdots \oplus M_n\}'' \subseteq \{M_1 \oplus \cdots \oplus M_n\}'$.

Let $X \in \{M_i\}'$. Then $X = M_\psi$, for some $\psi \in M(B_i)$, which implies that M_ψ commutes with M_ϕ for all $\phi \in M(B_i)$; thus $M_\psi \in \{M_\phi : \phi \in M(B_i)\}' = \{M_i\}''$. Therefore, $\{M_i\}'' = \{M_i\}'$, for $i = 1, \cdots, n$. Now,

$$\begin{aligned}\{M_1 \oplus \cdots \oplus M_n\}' &= \{M_1\}' \oplus \cdots \oplus \{M_n\}' \\ &= \{M_1\}'' \oplus \cdots \oplus \{M_n\}'' \\ &\subseteq \{\{M_1\}' \oplus \cdots \oplus \{M_n\}'\}' \\ &= \{\{M_1 \oplus \cdots \oplus M_n\}'\}' \\ &= \{M_1 \oplus \cdots \oplus M_n\}''\end{aligned}$$

Lemma 1. Suppose that $G_1 \cap G_2 = \varnothing$, and X and Y are operators in $L(B_2, B_1)$ such that $M_1 X - X M_2 = M_1 Y - Y M_2$. Then $X = Y$. In particular, if X intertwines M_1 and M_2 then $X = 0$.

Proof. Take an arbitrary $\lambda \in G_1$. From the hypothesis, it follows that

$$X^* M_1^* e_\lambda - M_2^* X^* e_\lambda = Y^* M_1^* e_\lambda - M_2^* Y^* e_\lambda;$$

or equivalently,

$$(\lambda - M_2^*)(X^* e_\lambda) = (\lambda - M_2^*)(Y^* e_\lambda).$$

This along with the invertibility of $\lambda - M_2^*$ imply that $X^* e_\lambda = Y^* e_\lambda$. But since the linear span of $\{e_\lambda : \lambda \in G_1\}$ is dense in B_1^*, it follows that $X = Y$.

An operator T acting on the Banach space $B_1 \oplus \cdots \oplus B_n$ is called *upper triangular*, if $T_{ij} = 0$ for $i > j$; that is, every entry below the main diagonal is zero. Furthermore, T is called *lower triangular* if $T_{ij} = 0$ for $i < j$; that is every entry above the main diagonal is zero. In the next result, we investigate when the direct sum of certain upper triangular and lower triangular operators split.

Theorem 2. Suppose that $G_1 \cap G_2 = \varnothing$. Let T and S be upper or lower triangular operators, respectively, on $\underbrace{B_1 \oplus \cdots \oplus B_1}_{n-times}$ and $\underbrace{B_2 \oplus \cdots \oplus B_2}_{n-times}$ with diagonals $T_{ii} = M_1$, and $S_{ii} = M_2, i = 1, \cdots, n$. Then $\{T \oplus S\}' = \{T\}' \oplus \{S\}'$.

Proof. We only prove the theorem, where T and S are both upper triangular operators. The other cases are proved by similar arguments.

Obviously, $\{T\}' \oplus \{S\}' \subseteq \{T \oplus S\}'$. To show that $\{T \oplus S\}' \subseteq \{T\}' \oplus \{S\}'$, let $X = \begin{bmatrix} C & E \\ F & D \end{bmatrix}$ commutes with $T \oplus S$; that is

$$\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} C & E \\ F & D \end{bmatrix} = \begin{bmatrix} C & E \\ F & D \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}$$

Then $TC = CT, TE = ES, SF = FT$, and $SD = DS$. So it is sufficient to show that E and F are the zero operators. Since $TE = ES$, by the matrix representation of operators, we have

$$\begin{bmatrix} M_1 & T_{12} & T_{13} & \cdots & T_{1n} \\ 0 & M_1 & T_{23} & \cdots & T_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_1 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ \vdots & \vdots & & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ \vdots & \vdots & & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{bmatrix} \begin{bmatrix} M_2 & S_{12} & S_{13} & \cdots & S_{1n} \\ 0 & M_2 & S_{23} & \cdots & S_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_2 \end{bmatrix};$$

so the following relations are obtained:

$$\begin{aligned} (TE)_{n1} &= M_1 E_{n1} = (ES)_{n1} = E_{n1} M_2, \\ (TE)_{n2} &= M_1 E_{n2} = (ES)_{n2} = E_{n1} S_{12} + E_{n2} M_2 \\ &\vdots \\ (TE)_{nn} &= M_1 E_{nn} = \sum_{i=1}^{n-1} E_{ni} S_{in} + E_{nn} M_2. \end{aligned}$$

Then, applying the preceding lemma, we conclude that $E_{ni} = 0$ for $i = 1, \cdots, n$. The second step is to show that $E_{(n-1)i} = 0$, for $i = 1, \cdots, n$. Similar to the above computations, we obtain

$$(TE)_{(n-1)1} = M_1E_{(n-1)1} + T_{(n-1)n}E_{n1} = (ES)_{(n-1)1} = E_{(n-1)1}M_2,$$
$$(TE)_{(n-1)2} = M_1E_{(n-1)2} + T_{(n-1)n}E_{n2} = E_{(n-1)1}S_{12} + E_{(n-1)2}M_2$$
$$\vdots$$
$$(TE)_{(n-1)n} = M_1E_{(n-1)n} + T_{(n-1)n}E_{n2} = \sum_{i=1}^{n-1} E_{(n-1)i}S_{in} + E_{(n-1)n}M_2.$$

So, again applying the preceding lemma, we observe that the $(n-1)th$ row of E is zero. Continuing the above process, every row of the matrix representation of E and so E becomes zero.

Since $SF = FT$, by the same way, we can show that $F = 0$.

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