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A STRUCTURE OF SEMI MODULE OVER A BOOLEAN LIKE SEMI RING

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ABSTRACT

In this paper, the concepts invariant R- sub groups of R and semi module over a Boolean like semi ring R are introduced and also study some of its properties. Further if R is weak commutative Boolean like semi ring and P is an R- sub module of M then (P: M) is invariant R- sub group of R. Also annihilator of a sub set P of M in R is a right ideal of R and if P is an R- sub module of M then Ann (P) is an ideal of R are proved.

Mathematics Subject Classification: *16Y30, 16 Y 60.*

Key Words: *Boolean like semi ring, Quotient Boolean like semi ring, Semi module, Quotient semi Module.*

INTRODUCTION

The concept of Boolean like semi rings is due to Venkateswarlu. K, Murthy B.V.N and Amaranth N [3]. It is well known that every ring is module over itself. In a similar manner, It is observed in definition 1.1 [3] that an abelian group (R,+) over itself satisfies a(x+y) = ax + ay and a(xy) = (ax)y. This idea is extended to any abelian group M over Boolean like semi ring R. The present paper is divided into 3 sections. In section 1, the preliminary concepts and results regarding Boolean like semi rings. In section 2, Invariant R-sub group of R is defined in Boolean like semi ring and also furnish examples(see 2.4.A,B,C).The concept of semi module is introduced (see definition 2.6) and also furnish examples (see example 2.8. A, B…I).Further (P:M) is defined and obtain that if R is weak commutative Boolean like semi ring and P is R-sub module of M then (P:M) is R-sub group of R (see corollary2.18.) and(P:M) is an ideal of R (see Theorem2.19.). In the last section certain properties of annihilators are obtained. Finally end this section with the theorem that Let M be an R- semi module, H an R - sub module of M and K an R-ideal of M then H + K is an R- sub module of M(see theorem 3.5.). Throughout this paper R is Boolean like semi ring and M is semi module over R.

1. PRELIMINARIES

We recall certain definitions and results concerning Boolean like semi rings from [3]

Definition 1.1: A non-empty set R together with two binary operations + and . satisfying the following conditions is called a Boolean like semi ring

1. (R, +) is an abelian group
2. (R, .) is a semi group
3. a.(b+c) = a.b +a.c for all a, b, c $\in$ R
4. a + a = 0 for all a $\in$ R
5. ab (a+b+ab) = ab for all a, b $\in$ R.

Let R be a Boolean like semi ring. Then

Lemma 1.2: For a $\in$ R, a.0 =0

Lemma 1.3: For a $\in$ R, $a^4 = a^2$ (weak idempotent law)

Remark 1.4: If R is a Boolean like semi ring then, $a^n = a$ or a^2 or a^3 for any integer n > 0

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Definition 1.5: A Boolean like semi ring R is said to be weak commutative if abc = acb, for all a, b, c$\in$ R.

Lemma 1.7: If R is a Boolean like semi ring with weak commutative then 0.a = 0, for all a$\in$R

Lemma 1.8: Let R be Boolean like semi ring then for any a, b$\in$ R and for any integers m, n > 0,

1. $a^m a^n = a^{m+n}$
2. $(a^m)^n = a^{mn}$
3. $(ab)^n = a^n b^n$ if R is weak commutative.

Definition 1.9: A non empty subset I of R is said to be an ideal if

1. (I,+) is a sub group of (R,+) ,i.e. for a, b$\in$ R $\Rightarrow$ a + b $\in$ R
2. ra$\in$ R for all a$\in$ I , r $\in$R ,i.e. RI $\subseteq$ I
3. (r+a) s + rs$\in$ I. for all r, s $\in$R , a $\in$ I

Remark1.10: If I satisfies 1 and 2, I is called left ideal and If I satisfies 1 and 3, I is called right ideal of R.

Remark 1.11: If R is weak commutative Boolean like semi ring then ar$\in$I for all a$\in$I and r$\in$R.

Definition 1.12: An element 1$\in$ R is said to be unity if a1=1a = a, for all a$\in$R. If a1 = a, then 1 is called right unity and if 1a = a, then 1 is called left unity.

Theorem 1.13: Let R be a Boolean like semi ring with unity 1. If I is an Ideal of R such that 1 $\in$ I then I = R.

2 INVARIANT SUB GROUPS AND SEMI MODULES

Definition 2.1: A sub set H of R is called (two sided or invariant) R – subgroup of R if

(a) (H , +) is a sub group of (R ,+) (b)RH$\subseteq$ H (c) HR$\subseteq$ H

Remark 2.2: In the above definition H satisfies (a) and (b), H is called left R – sub group of R and H satisfies (a) and (c), H is called right R – sub group of R.

Theorem 2.3: If a $\in$ R then aR is a right R – sub group of R.

Proof: Let ar, as $\in$aR Then ar + as = a (r +s) $\in$aR

Hence aR is a sub group of R.

Now (aR) R = a(RR) $\subseteq$aR. Thus aR is a right R – sub group of R.

Example 2.4:

A) Let R = {0, a, b, c}. The binary operations + and. are defined as follows

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	a	b	c

Then (R, +,.) is a Boolean like semi ring. We observe that cab $\neq$ cba.

Clearly H = {0, b} is a right R - sub group of R.

H = {0, a} is R - sub group of R.

H = {0, c} is neither right nor left R - sub goup of R.

B) Let R = {0, x, y, z}. The binary operations + and. are defined as follows

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

.	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Then (R, +, .) is a Boolean like semi ring. We note that abc = acb, for all a, b, c∈ R.

Clearly {0, x} and {0, z} both are right R – sub groups of R and also {0, z} is R – sub group of R.

C) Let R = {0, p, q, 1}. The binary operations + and . are defined as follows

+	0	p	q	1
0	0	p	q	1
p	p	0	1	q
q	q	1	0	p
1	1	q	p	0

.	0	p	q	1
0	0	0	0	0
p	0	0	p	p
q	0	0	q	q
1	0	p	q	1

Then R is a Boolean like semi ring. It is clear that a.1 = 1.a = a for all a ∈R.

Clearly H = {0, q} is a right R - sub group of R.

H = {0, p} is R - sub group of R.

H = {0, 1} is neither right nor left R - sub goup of R.

MODULES:

Definition 2.6: Let R be a Boolean like semi ring and (M, +) be an abelian group then M is called a semi R – module if there is a mapping . : M x R → M (the image of (m,r) under th mapping is denoted by mr) such that ‘

m(r+s) = mr + ms and m(rs) = (mr)s, for all m∈ M, r, s∈ R.

Remark 2.7: If R is Boolean like semi ring then obviously (R, +) is itself R- Module.

Examples 2.8:

A) Let R = { 0,a,b,c}, see example 2.4(A) and M = {0,1,2,3} is abelian group under the binary operation "+" addition modulo 4 is defined and define * :M x R→ M as follows

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

*	0	a	b	c
0	0	0	0	0
1	0	0	2	2
2	0	0	2	2
3	0	0	0	0

Then M is semi module over R. We observed that Chr M $\neq 2$.

B) Let R = { 0,a,b,c}, see example 2.4(A) and M = { 0,1,2,3} is abelian group under the binary operation "+" addition modulo 4 is defined and define * :M x R → M as follows

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

*	0	a	b	c
0	0	0	0	0
1	0	0	2	2
2	0	0	2	2
3	0	0	2	2

Then M is semi module over R. We observed that Chr M $\neq 2$.

C) Let R = {0,a,b,c}, see example 2.4(A) and M = {0,1,2,3} is abelian group under the binary operation + is defined and define * :M x R $\rightarrow$ M as follows

+	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	1
3	3	0	1	0

*	0	a	b	c
0	0	0	0	0
1	0	0	2	2
2	0	0	2	2
3	0	0	2	2

Then M is semimodule over R. We observed that Chr M = 2.

D) Let R = {0,a,b,c}, see example 2.4(A) and M = {0,1,2,3} is abelian group under the binary operation + is defined and define * :M x R $\rightarrow$ M as follows

+	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	1
3	3	0	1	0

*	0	a	b	c
0	0	0	0	0
1	0	0	3	3
2	0	0	3	3
3	0	0	3	3

Then M is semimodule over R. We observed that Chr M = 2.

E) Let R = {0,a,b,c}, see example 2.4(A) and M = {0,1,2,3} is abelian group under the binary operation + is defined and define * :M x R $\rightarrow$ M as follows

+	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	1
3	3	0	1	0

*	0	a	b	c
0	0	0	0	0
1	0	0	2	2
2	0	0	2	2
3	0	0	3	3

Then M is semimodule over R. We observed that Chr M = 2.

F) Let R = { 0,a,b,c}, see example 2.4(A) and M = { 0,1,2,3} is abelian group under the binary operation + is defined and define * :M x R $\rightarrow$ M as follows

+	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	1
3	3	0	1	0

*	0	a	b	c
0	0	0	0	0
1	0	0	2	2
2	0	0	3	3
3	0	0	3	3

Then M is semimodule over R. We observed that Chr M = 2.

G) Let R ={0, p, q,1}, see example 2.4(C) and M = {0,1,2,3} is abelian group under the binary operation + is defined and define * :M x R $\rightarrow$ M as follows

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	0	1	0

*	0	p	q	1
0	0	0	0	0
1	0	0	3	3
2	0	0	2	2
3	0	0	3	3

Then M is semimodule over R. We observed that Chr M = 2 and $m*1 \neq m$.

H) Let R ={0,p,q,1}, see example 2.4(C) and M = {0,1,2,3} is abelian group under the binary operation + is defined and define * :M x R $\rightarrow$ M as follows

+	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	1
3	3	0	1	0

*	0	p	q	1
0	0	0	0	0
1	0	0	1	1
2	0	0	2	2
3	0	0	3	3

Then M is semimodule over R. We observed that Chr M = 2 and $m*1= m$.

I) Let R = { 0,p,q,1}, see example 2.4(C) and M = {0,1,2,3} is abelian group under the binary operation "+" addition modulo 4 is defined and define * :M x R $\rightarrow$ M as follows

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

*	0	p	q	1
0	0	0	0	0
1	0	0	2	2
2	0	0	2	2
3	0	0	0	0

Then M is semimodule over R. We observed that Chr M $\neq$ 2and $m*1\neq m$.

Definition2.8: Let H be a sub group of M such that for all r $\in$R, for all h $\in$ H, we have that hr$\in$ H then H is called R – sub module of M , we denote $H <_R M$

Definition2.9: If R is weak commutative and M is module over R, 0$\in$M, 0r = 0, for all r$\in$ R.

Theorem 2.10: If M is semi module over R then m0 = 0 for all m $\in$M

Theorem2.11: If M is an R-module and m$\in$M then mR is an R – submodule of M.

Proof: mR = {mr / r $\in$R}. Clearly mR $\subseteq$ M.

Let mr, ms $\in$ mR where r, s$\in$ R, then mr-ms = m(r – s)$\in$mR

Hence mR is a sub group of M.

Let h $\in$ mR then h = mr , for some r$\in$ R. Now for all s$\in$ R, hs = (mr)s = m(rs)$\in$mR

Hence mR is a sub module of M over R.

Definition2.12: A sub group P of a module M is called R-ideal of M if for all $r \in R$ $m \in M$ and $n \in P$,we have $(m+n)r - mr \in P$.

Remark2.13: If M = R then R – ideals of M becomes right ideals of R and the R – sub modules of M are the right R – sub groups of R.

Example2.14: Let R = {0, a, b, c}. H = {0, b} and K = {0, a} both are proper R –sub modules. Also H = {0, b} is not an R- ideal, since (a+b)b – ab = c, which is not in H.

From this example, we say that R – sub modules of M are not necessarly R- ideals of M.

Theorem2.15: If R is weak commutative then every R- ideal of M is an R- sub module of M.

Proof: Let P be an R -ideal of M then P is sub group of M. We prove P is a sub module of M. It is enough to show that $r \in R$, $p \in P$ implies that $pr \in P$.

Write $pr = (0 + p)r - 0r \in P$

Definition2.16: Let P be an R -ideal of M then $(P:M) = \{r \in R \,/\, Mr \subseteq P\}$

Theorem2.17. If P is an R-sub module of M then (a) (P: M) is a right R-sub group of R. (b) (P: M) is a left R-sub group of R if R is weak commutative.

Proof: $(P: M) = \{r \in R \,/\, Mr \subseteq P\}$.

(a) First we prove (P: M) is sub group of (R, +)

For $r,s \in (P:M)$ then $Mr \subseteq P$,$Ms \subseteq P$

$M(r+s) \subseteq Mr+Ms \subseteq P$. Hence $r+s \in (P: M)$

Thus (P: M) is a sub group of R

Now we Prove $(P:M)R \subseteq (P:M)$

Let $r \in R$, $p \in (P:M)$ then $Mp \subseteq P$
$M(pr) = (Mp)r \subseteq Pr \subseteq P$ (since P is sub module), hence $pr \in (P:M)$.

Thus (P:M) is a right R- sub group of R.

(b) In (a), (P: M) is sub group of (R, +). Finally we show that $R(P:M) \subseteq (P:M)$

Let $r \in R$,$p \in (P:M)$ then $Mp \subseteq P$

Consider $Mrp = M\,rp(r + p + rp) = Mrpr + Mrpp + Mrprp$

$$= Mrrp + Mrrp + Mrrpp \subseteq Mp + Mp + Mp \subseteq P + P + P \subseteq P$$

Hence $Mrp \subseteq P$, $rp \in P$ thus (P: M) is a left R-sub group of R.

Corollary 2.18: If R is weak commutative Boolean like semi ring and P is an R-sub module of M then (P : M) is invariant R-sub group of R.

Theorem2.19: If P is an R- ideal of M then (P:M) is an ideal of R.

Proof: If P is an R- ideal of M then P is sub group of M and for all$r \in R$, for all $n \in P$, for all $m \in M$, we have $(m+n)r-mr \in P$

If P is an R- ideal of M then P is R – submodule of M, i.e $PR \subseteq P$

We prove (P: M) is an ideal of R.

(i) For a, b∈ (P:M)⇒ Ma, Mb ⊆ P ⇒Ma+Mb ⊆ P ⇒M(a+b) ⊆ P ⇒ a +b ∈ (P:M)
(ii) For a ∈ (P:M) , r∈ R ⇒ Ma ⊆ P

Since M is R – module, Hence Mr ⊆ M

Mr ⊆ M ⇒Mra ⊆ Ma ⊆ P⇒ra∈(P:M)

(iii) For r, s∈ R , a ∈ (P:M) Ma ⊆ P

For all m ∈M, m[(r+a)s –rs] = m (r+a) s – mrs = (mr+ma)s-mrs∈ P

Hence (r+a)s – rs∈ (P:M)

Thus (P: M) is an ideal of R.

Proposition 2.20: Let I be an ideal of R then R/I is semi module over R with scalar multiplication defined by:

(s+I)r = sr +I for all r, s ∈ R.

Proof: Let r, s, t ∈ R

(i) (r + I) (s+t) = r(s+t) + I = (rs+rt) +I = (rs+I) + (rt+I) = (r+I)s + (r+I)t
(ii) (r+I)st = r(st) +I = (rs)t + I = (rs + I)t = ((r + I)s)t

Theorem 2.21: Let M be a semi module over R and P be an R- ideal of M. Then the quotient group M/P = {m + P / m∈M }is semi module over R (called the quotient semi module over R) with scalar multiplication defined by (m + P) r = mr + P , for all r ∈R, m∈M.

Proof: Same as the proof of proposition 2.20.

3 ANNIHILATORS

Definition 3.1: If P ⊆ M then annihilator of P in R is defined by Ann (P) = {r∈R/Pr = {0}}

Theorem 3.2: If M is an R-module and P ⊆ M then

(i) Ann(P) is a right ideal of R
(ii) If P is an R-sub module of M then Ann(P) is an ideal of R.

Proof:

(i) Let r, s ∈R such that r, s∈Ann(P) then Pr = {0} = Ps

Now for all p ∈P, p(r+s) = pr + ps = 0 + 0 = 0, Hence x + y ∈Ann(P)

Let r, s ∈ R , x ∈ Ann(P) then Px = {0}

Now for all p ∈P p[(r+ x)s + rs] = p[(r+ x)s - rs]

= p (r+ x)s -prs =(pr+px)s – prs

=(pr+0)s – prs = prs – prs = 0

Hence [(r+ x)s + rs] ∈Ann(P) , Thus Ann (P) is a right ideal of R.

(ii) From (i) , If P is an R-sub module of M then Ann(P) is a right ideal of R. It is sufficient to prove that for all r∈R , a ∈Ann(P)⇒ra∈Ann(P)

Let r∈R , a ∈Ann(P) ⇒Pa = {0}

Now $P(ra) = (Pr)a \subseteq Pa = \{0\} \Rightarrow ra \in Ann(P)$

Thus Ann (P) is an ideal of R

Proposition3.3:

(a) If M is an R- semi module and $I \subseteq Ann(M)$ then M is an R/I – semi module with respect to $m(r + I) = mr$.

(b) If M is an R/I –semi module then M becomes an R- semi module under the scalar multiplication $mr = m(r + I)$, with $I \subseteq Ann(M)$

Proof:

(a) Define a map $M \times R/I \to M$ as $m(r + I) = mr$, for all $m \in M$, $r \in R$

Let r,s R such that $r+ I = s + I \Rightarrow r+s \in I \subseteq Ann(M) \Rightarrow M(r+s) = 0 \Rightarrow m(r+s) = 0$,for all $m \in M \Rightarrow m(r-s) = 0 \Rightarrow mr - ms = 0 \Rightarrow mr = ms \Rightarrow m(r+I) = m(s+I)$

Hence the given map is well defined

(i) $m[(r+I)+(s+I)] = m[(r+s)+I] = m(r+s) = mr + ms = m(r+I) + (s+I)$

(ii) $m[(r+I) (s+I)]=m [(rs+I)] = m(rs) = (mr)s = mr(s+I) = [m(r+I)](s+I)$

Thus M is R/I – Module

(b) Proof follows the reverse process in (a).

Further more, if $x \in I \Rightarrow x + I = 0 + I \Rightarrow m (x + I) = m(0+I) =0$, for all $m \in M$

$\Rightarrow mx = 0 \Rightarrow x \in Ann(M)$, hence $I \subseteq Ann(M)$

Theorem3.4: Let M be an R-semimodule, H an R – sub module of M and K an R-sub module (R-ideal) of M then $H \cap K$ is an R- sub module (R-ideal) of M

Proof: Proof is routine verification.

Theorem3.5: Let M be an R- semi module, H an R - sub module of M and K an R-ideal of M then H + K is an R- sub module of M.

Proof: $H+K = \{h + k / h \in H, k \in K\}$

Clearly H+K is a sub group of M.

(i) Now we prove for all $x \in H + K$,$r \in R \Rightarrow x r \in H+K$

For $x \in H+K \Rightarrow x = h+k$,$h \in H$, $k \in K$

$xr = (h+k)r = (h+k)r - hr + hr = [(h+k)r - hr] + hr \in K+H$

REFERENCES

[1] Gunter Pilz: Near-Rings, The theory and its applications (North-Holland) 1983.

[2] Mr. S. Juglal: Prime near ring modules and their links with the generalized group near -ring

[3] Venkateswarlu. K and Murthy B.V.N and Amaranth N: Boolean like semi rings, Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 13, 619 – 635

[4] Venkateswarlu. K and Murthy B.V.N: primary ideals in a Boolean like semi rings. (Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 28, 1367 – 1377

[5] Venkateswarlu. K and Murthy B.V.N: Semi prime ideals and Annihilators in Boolean like semi rings, International Journal of Algebra, Vol. 5, 2011, no. 28, 1363 – 1370.

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