**International Journal of Mathematical Archive-2(3), Mar. - 2011, Page: 310-314**
**Available online through www.ijma.info** **ISSN 2229 – 5046**

# FIXED POINT THEOREMS FOR E-NONEXPANSIVE MAPPINGS

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*(Received on: 17-02-11; Accepted on: 27-02-11)*

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***ABSTRACT***

*Youness introduced the concept of E-convex sets in $R^n$. Following this Sheiba Grace and Thangavelu discussed the algebraic properties of E-convex sets. Wataru Takahashi introduced a convex structure in metric spaces and formulated some fixed point theorems for nonexpansive mappings. The authors[6] introduced E-convex structure in metric spaces. The purpose of this paper is to formulate some fixed point theorems in E-convex metric spaces.*

***Key words**: Fixed points, E-convex metric spaces and E-nonexpansive mappings.*

***MSC 2010**: 47H10*

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## 1. INTRODUCTION AND PRELIMINARIES:

Youness[8] introduced the concept of E-convex sets in $R^n$. Sheiba Grace and Thangavelu[5] discussed the algebraic properties of E-convex sets. Wataru Takahashi[7] studied some fixed point theorems for nonexpansive mappings of convex metric spaces. The authors[6] introduced the concept of E-convex metric spaces. In this paper we discuss some fixed point theorems in E-convex metric spaces. We also extend some theorems and results of Wataru Takahashi [7] to E-convex metric spaces. We recall the following definitions and results.

**Definition: 1.1**
Let (X, d) be a metric space and I = [0, 1]. Let W: $X\times X\times I\to X$ be a mapping and E: $X\to X$ be a map. Then **(i)** W is a convex structure[7] on X if for each $(x, y; \lambda) \in X\times X\times I$ and $u\in X$, $d(u, W(x, y; \lambda)) \le \lambda d(u, x) + (1-\lambda)d(u, y)$ in which case the triplet (X, d, W) is called a convex metric space. **(ii)** W:$X\times X\times I\to X$ is an E-convex structure[6] on X if for each $(x, y; \lambda)\in X\times X\times I$ and $u\in X$, $d(E(u), W(x, y; \lambda))\le \lambda d(E(u), E(x))+ (1-\lambda)d(E(u), E(y))$ in which case the 4-tuple (X, d, W, E) is called an E-convex metric space.

**Definition: 1.2**
Let $M\subseteq X$. (i) M is a convex[7] subset of a convex metric space (X, d, W) if $W(x, y; \lambda)\in M$ for all $x, y\in M$ and $\lambda$ $(0\le\lambda\le1)$ and (ii) M is an E-convex[6] subset of an E-convex metric space (X, d, W, E) if $W (x, y; \lambda)\in M$ for all $x, y\in M$ and $\lambda$ $(0\le\lambda\le1)$.

**Definition: 1.3**
A convex metric space (X, d, W) is said to have the Property(C)[7] if every bounded decreasing sequence of nonempty closed convex subsets of (X, d, W) has a nonempty intersection.

**Definition: 1.4**
An E-convex metric space (X, d, W, E) has the Property ($C_E$) [6] if every bounded decreasing sequence of nonempty closed E-convex subsets of (X, d, W, E) has a nonempty intersection.

**Definition: 1.5**
Let A be a subset of (X, d, W). A point $x\in A$ is a diametral point [7] of A provided the diameter of A $=\delta(A) = \sup\{d(x, y): y\in A\}$.

**Definition: 1.6**
Let A be a subset of (X, d, W, E). A point $x\in A$ is an E-diametral point [6] of A provided the E-diameter of A $=\delta_E(A) = \sup\{d(E(x), E(y)): y\in A\}$.

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**Definition: 1.7**
A convex metric space (X, d, W) is said to have normal structure[7] if for each closed bounded convex subset A of (X, d, W) which contains at least two points, there exists $x\in A$ which is not a diametral point of A.

**Definition: 1.8**
An E-convex metric space (X, d, W, E) is said to have E-normal structure[6] if for each closed bounded E-convex subset A of (X, d, W, E) which contains at least two points, there exists $x\in A$ which is not an E- diametral point of A.

**Definition: 1.9**
A convex metric space (X, d, W) is said to be strictly convex [7] if for
any $x, y\in X$ and $\lambda$ ($0\leq\lambda\leq 1$), there exist a unique element $z\in X$ such that $\lambda d(x, y) = d(z, y)$ and $(1- \lambda)d(x, y) = d(x, z)$.

**Definition: 1.10**
Let $E: X\to X$ be a map and (X, d, W, E) be an E-convex metric space with EW=W. Then (X, d, W, E) is said to be strictly E-convex [6] if for any $x, y\in X$ and $\lambda(0\leq\lambda\leq 1)$, there exist a unique element $z\in X$ such that

$$\lambda d(E(x), E(y)) = d(E(z), E(y)) \text{ and } (1- \lambda)d(E(x), E(y)) = d(E(x), E(z)).$$

**Definition: 1.11**
Let (X, d, W) be a convex metric space and K be a subset of (X, d, W). A mapping T of K into X is said to be nonexpansive [7] if for each pair of elements x and y of K, we have $d(Tx, Ty) \leq d(x, y)$.

**Definition: 1.12**
Let (X, d, W, E) be an E-convex metric space and K be a subset of an E - convex metric space (X, d, W, E). A mapping T of K into X is said to be E- nonexpansive[6] if for each pair of elements x and y of K, we have $d(TE(x), TE(y)) \leq d(E(x), E(y))$.

Wataru Takahashi [7] used the following notations for a subset A of X.

$$S(x, r) =\{y\in X: d(x, y) < r\};$$
$$S[x, r] = \{y\in X: d(x, y) \leq r\};$$
$$R_x (A) = \sup\{d(x, y): y\in A\};$$
$$R(A) = \inf\{ R_x (A): x\in A\};$$
$$A_c = \{ x\in A : R_x(A) = R(A)\}.$$

**Lemma: 1.13 (**Proposition 4, [7]**)**
If (X, d, W) has Property(C), then $A_c$ is nonempty, closed and convex.

**Lemma: 1.14 (**Proposition 5, [7]**)**
Let M be a nonempty compact subset of (X, d, W) and let K be the least closed convex set containing M. If the diameter $\delta(M)$ is positive, then there exists an element $u\in K$ such that $\sup\{d(x, u): x\in M\} < \delta(M)$.

**Lemma: 1.15 (**Theorem 1, [7]**)**
Suppose that (X, d, W) has Property(C). Let K be a nonempty bounded closed convex subset of (X, d, W) with normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.

**Lemma: 1.16 (T**heorem 2, [7]**)**
Suppose (X, d, W) being strictly convex with Property(C). Let K be a nonempty bounded closed convex subset of (X, d, W) with normal structure. If $\mathscr{F}$ is a commuting family of nonexpansive mappings of K into itself, then the family has a common fixed point in K.

The authors [6] used the following notations for a subset A of X.

$$S_E(x, r) = \{y\in X: d(E(x), E(y)) < r\};$$

$$S_E[x, r] = \{y\in X: d(E(x), E(y)) \leq r\};$$

$$(R_x)_E (A) = \sup\{d(E(x), E(y)): y\in A\};$$

$$R_E(A) = \inf\{ (R_x)_E (A): x\in A\};$$

$$(A_c)_E = \{x\in A : (R_x)_E (A) = R_E(A)\}.$$

**Lemma: 1.17** (Theorem 2.11, [6])

Let E: $X\to X$ be a map and (X, d, W, E) be an E-convex metric space with EW = W. If (X, d, W, E) has the Property (EC), then $(A_c)_E$ is nonempty closed and convex.

## 2. PROPERTIES:

In this section we discuss some properties of E-convex metric spaces that will be useful in sequel. Let E: $X\to X$ be an idempotent map. Let (X, d) be a metric space .Then (EX, d) is a metric subspace of (X, d). Suppose W: $X\times X\times I\to X$ is an E-convex structure of (X, d) with the property that W maps the elements of $EX \times EX \times I$ to the elements of EX. Then

$$\begin{aligned} d(E(u), W(E(x), E(y); \lambda)) &\le \lambda d(E(u), E^2(x)) + (1-\lambda)d(E(u), E^2(y)) \\ &= \lambda d(E(u), E(x)) + (1-\lambda)d(E(u), E(y)). \end{aligned}$$

Therefore the triplet (EX, d, $W_E$) is a convex metric space defined by $W_E(E(x), E(y);\lambda) = W(E(x), E(y);\lambda)$ for all x, $y\in X$.

**Proposition: 2.1**

Suppose E is idempotent, injective with $EW(x, y;\lambda) = W(E(x), E(y);\lambda)$ where $W_E$ is defined as before. Let $A \subseteq X$. Then A is E-convex in (X, d, W, E) if and only if EA is convex in (EX, d, $W_E$).

**Proof:** Suppose A is E-convex in (X, d, W, E). Let $E(x), E(y)\in EA$ and $\lambda$ $(0 \le \lambda\le 1)$. Since E is injective, $x, y\in A$. Since A is E-convex in (X, d, W, E), by Definition 1.2, $W(x, y;\lambda)\in A$ that is $E(W(x, y;\lambda))\in EA$. Since $EW(x, y;\lambda) = W(E(x), E(y);\lambda)$, $W(E(x), E(y);\lambda) \in EA$. This shows that EA is convex in (EX, d, $W_E$).

Conversely, assume that EA is convex in (EX, d, $W_E$). Now let $x, y\in A$ and $\lambda$ $(0 \le \lambda\le 1)$. Then $E(x), E(y)\in EA$. Since EA is convex in EX, by Definition 1.2, $W(E(x), E(y);\lambda)\in EA$. Again since $EW(x, y;\lambda) = W(E(x), E(y);\lambda)$, $E(W(x, y;\lambda))\in EA$. Since E is one-one, $W(x, y;\lambda) \in A$. This shows that A is E-convex in (X, d, W, E). This completes the proof.

The next Lemma gives the relationships between the notations used in[6] and the notations used in [7].

**Lemma: 2.2**

Let (X,d) be a metric space. Suppose E: $X\to X$ is injective. Then for any subset A of X

(i) $R_x(E(A)) = (R_x)_E(A)$;
(ii) $R(E(A)) = R_E(A)$;
(iii) $E(A_c) = (A_c)_E$, provided E(A) = A;
(iv) $\delta(E(A)) = \delta_E(A)$.

**Proof:** $R_x(E(A)) = \sup\{ d(E(x), E(y)): E(y)\in EA\}= \sup\{d(E(x), E(y)): y\in A\}= (R_x)_E(A)$.
$R(E(A)) = \inf\{ R_x(E(A)): E(x)\in EA\} = \inf\{ (R_x)_E(A): x\in A\} = R_E(A)$.
$$\begin{aligned} E(A_c) &= \{ E(x)\in EA: R_{Ex}(E(A)) = R(E(A))\} = \{ E(x)\in EA : (R_{Ex})_E(A) = R_E(A)\} \\ &= \{y\in EA: (R_y)_E(A) = R_E(A)\} = (A_c)_E. \end{aligned}$$
$\delta(E(A)) = \sup\{d(E(x), E(y)): E(x), E(y)\in EA\} = \sup\{d(E(x), E(y)): x, y\in A\} = \delta_E(A)$.

**Proposition: 2.3**

Suppose E is idempotent with $EW(x, y;\lambda) = W(E(x), E(y);\lambda)$ where $W_E$ is defined as before. Then

**(i)** $E(S_E(x, r)) = S(E(x), r)$;
**(ii)** $E(S_E[x, r]) = S[E(x), r]$.

**Proof:** Let $z\in E(S_E(x, r))$ with $z = E(y)$ for some $y\in S_E(x, r)$. Then $d(E(x), E(y))< r$. This implies $z = E(y)\in S(E(x), r)$. Conversely let $z\in S(E(x), r)$. Then $d(E(x), z)<r$ and $z\in EX$. Therefore $z = E(y)$ for some $y\in X$ that implies $d(E(x), E(y)) < r$. This shows $y\in S_E(x, r)$ that implies $z\in E(S_E(x, r))$. This shows that $E(S_E(x, r)) = S(E(x), r)$. This completes the proof for (i) and the proof for (ii) is analog.

**Proposition: 2.4**

Suppose E is idempotent with $EW(x, y;\lambda) =W(E(x), E(y);\lambda)$ where $W_E$ is defined as before. Let (X, d, W, E) be an E-convex metric space. Then for $x, y\in X$,

$$d(E(x), E(y)) = d(E(x),W(E(x), E(y); \lambda))+d((W(E(x),E(y); \lambda), E(y)), \text{ for } 0 \le \lambda \le 1.$$

**Proof:** Let $x, y\in X$. Then $E(x), E(y)\in EX$. Since (EX, d, $W_E$) is a convex metric space

 

$d(E(x), E(y)) = d(E(x), W_E(E(x), E(y); \lambda)) + d((W_E(E(x), E(y); \lambda), E(y))$
$= d (E(x), W (E(x), E(y); \lambda)) + d((W(E(x), E(y); \lambda), E(y)).$

**Proposition: 2.5**
Let A be a subset of X. Suppose E is idempotent, injective, E(A)=A. Suppose (EX, d, $W_E$) has the property(C) and E is a closed map. Then E($A_c$) is nonempty, closed and convex in (EX, d, $W_E$).

**Proof:** By Lemma 1.17, $(A_c)_E$ is nonempty, closed and E-convex in (X, d, W, E). By Lemma 2.2 $E(A_c) = (A_c)_E$ that implies $E(A_c)$ is nonempty closed and E-convex in (X, d, W, E).
Now by using Proposition 2.1, $E(E(A_c))$ is convex in (E(X), d, $W_E$). Since E is idempotent, $E(A_c)$ is nonempty and convex in (E(X), d, $W_E$). Since E is a closed map, $E(A_c)$ is nonempty, closed and convex (E(X), d, $W_E$). This completes the proof.

**Proposition: 2.6**
Let E be idempotent and injective. Let M be a non empty compact subset of (E(X), d, $W_E$) and let K be the least closed convex set containing M. If the diameter $\delta(M)$ is positive, then there exists an element $u \in K$ such that sup $\{d(x, u): x \in M\} < \delta(M)$.

**Proof:** Since (EX, d, $W_E$) is a convex metric space, the proof follows from Lemma 1.14.

## 3. FIXED POINT THEOREMS:

**Lemma: 3.1**
Suppose E is idempotent, injective with $EW(x, y; \lambda) = W(E(x), E(y); \lambda)$ where $W_E$ is defined as in section 2. Suppose the map E satisfies the property that for every bounded closed subset B of E(X), there exists a bounded closed subset A of X with E(A) =B. If (X, d, W, E) has E-normal structure, then (E(X), d, $W_E$) has normal structure.

**Proof:** Suppose the E-convex metric space (X, d, W, E) has E-normal structure. Clearly (E(X), d, $W_E$) is a convex metric space. Let B be a closed bounded convex subset of E(X), containing at least two points. Then B = E(A) for some bounded closed subset A of X. Since B is convex in (E(X), d, $W_E$), by Proposition 2.1, A is E-convex in (X, d, W, E). This shows that A is closed bounded and E-convex in (X, d, W, E). Since (X, d, W, E) has E-normal structure, by Definition 1.8, there exists $y \in A$ such that y is not an E-diametral point of A. Since $\delta(E(A)) = \delta_E(A)$, $\sup\{d(E(x), E(y)): x \in A\} \neq \delta(E(A))$ that is $E(y) \in EA$ is not a diametral point of E(A). Therefore (E(X), d, $W_E$) has normal structure. This completes the proof.

**Lemma: 3.2**
Suppose E is idempotent, injective with $EW(x, y; \lambda) = W(E(x), E(y); \lambda)$ where $W_E$ is defined as in section 2. Suppose the map E satisfies the property that for every bounded closed subset B of E(X), there exists a bounded closed subset A of X with E(A)=B. If (X, d, W, E) has Property ($C_E$), then (E(X), d, $W_E$) has Property(C).

**Proof:** Let $B_1 \supseteq B_2 \supseteq \ldots$ be a decreasing sequence of nonempty, bounded closed convex subsets of E(X). By the assumption there exists a nonempty bounded closed subsets of $A_1, A_2, \ldots$ of X such that $E(A_i) = B_i$ for every $i=1,2\ldots$. Since E is injective, $A_1 \supseteq A_2 \supseteq \ldots$ is a decreasing sequence of nonempty, bounded closed subsets of X. Since (X, d, W, E) has the Property ($C_E$), $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$ that is $E(\bigcap_{i=1}^{\infty} A_i) \neq \emptyset$. Since $E(\bigcap_{i=1}^{\infty} A_i) \subseteq \bigcap_{i=1}^{\infty} EA_i = \bigcap_{i=1}^{\infty} \mathbf{B}_i$. Now $\bigcap_{i=1}^{\infty} \mathbf{B}_i \neq \emptyset$. This shows that (E(X), d, $W_E$) has Property(C).

**Lemma: 3.3**
Suppose E is idempotent, injective with $EW(x, y; \lambda) = W(E(x), E(y); \lambda)$ where $W_E$ is defined as in section 2 and E(K) = K for some $K \subseteq X$. If T is an E-nonexpansive mapping of K into itself, then T is a nonexpansive mapping of K into itself.

**Proof:** If T is an E-nonexpansive mapping of K then for each pair of elements x, y of K, $d(TE(x), TE(y)) \leq d(E(x), E(y))$. Since $x, y \in K$, $E(x), E(y) \in E(K)$. Since E(K)=K, $E(x), E(y) \in K$ that implies $d(TE^2(x), TE^2(y)) \leq d(E^2(x), E^2(y))$ that is $d(TE(x), TE(y)) \leq d(E(x), E(y))$. This shows that T is a nonexpansive mapping of E (K) into E (K). Since E (K) = K, T is a nonexpansive mapping of K into itself. This completes the proof.

**Lemma: 3.5**
Suppose E is idempotent, injective with $EW(x, y; \lambda) = W (E(x), E(y); \lambda)$ where $W_E$ is defined as in section 2. If (X, d, W, E) is strictly E-convex, then (E(X), d, $W_E$) is strictly convex.

**Proof:** Suppose (X, d, W, E) is strictly E-convex. Then by Definition 1.10, for any $x, y \in X$ and $\lambda$, there exist a unique $z \in X$ such that $\lambda d(E(x), E(y)) = d(EW(x, y; \lambda), E(y))$ and $(1-\lambda)d(E(x), E(y)) = d(E(x), EW(x, y; \lambda))$. Since E is injective x,

 

$y \in X$, $E(x), E(y) \in E(X)$ and $\lambda$ such that

$$\lambda d(E^2(x), E^2(y)) = d(W(E(x), E(y);\lambda), E^2(y))$$

and

$$(1-\lambda)d(E^2(x), E^2(y)) = d(E^2(x), EW(E(x), E(y);\lambda))$$

that is

$$\lambda d(E(x),E(y)) = d(W(E^2(x),E^2(y);\lambda),E(y))$$

and

$$(1-\lambda)d(E(x),E(y)) = d(E(x),W(E^2(x),E^2(y);\lambda)).$$

Taking $E(z) = W(E^2(x),E^2(y);\lambda)$ it follows that $E(z) \in EX$. This shows that $(E(X), d, W_E)$ is strictly convex. This completes the proof.

**Theorem: 3.5**

Suppose E is idempotent, injective with $EW(x, y;\lambda) = W(E(x), E(y);\lambda)$ where $W_E$ is defined as in section 2 and (X, d, W, E) has Property ($C_E$). Let E be a map such that for every bounded closed subset B of E(X), there exists a bounded closed subset A of X with E(A) =B. Let K be an E-convex subset of (X, d, W, E) with E-normal structure and E(K) = K for some $K \subseteq X$. If T is an E-nonexpansive mapping of K into itself, then T has a fixed point in K.

**Proof:** Suppose T is E-nonexpansive mapping of K. Clearly $(E(X), d, W_E)$ is a convex metric space. Since (X, d, W, E) has Property($C_E$) then by Lemma 3.2 $(E(X), d, W_E)$ has Property(C). Since K is an E-convex subset of (X, d, W, E) with E-normal structure then by Lemma 3.1 $(E(X), d, W_E)$ is a normal structure. If T is an E-nonexpansive mapping in (X, d, W, E), then by Lemma 3.3 T is a nonexpansive mapping in $(E(X), d, W_E)$. Now to prove T has a fixed point in E(K). T is a map from T(K) to T(K) . Let $y \in E(K)$ with T(y) = y for some $y=E(x) \in E(K)$. Since E(K)= K shows T has a fixed point in K. This completes the proof.

**Theorem: 3.6**

Suppose E is idempotent, injective with $EW(x, y;\lambda) = W(E(x), E(y);\lambda)$ where $W_E$ is defined as in section 2. Let E be a map such that for every bounded closed subset B of E(X), there exists a bounded closed subset A of X with E(A) =B. Suppose (X, d, W, E) is strictly E-convex with Property ($C_E$). Let K be a nonempty bounded closed subset of (X, d, W, E) with E-normal structure and E(K) = K for some $K \subseteq X$. If $\mathscr{F}$ is a commuting family of E-nonexpansive mappings of K into itself, then the family has a common fixed point in K.

**Proof:** If T is an E-nonexpansive mapping in (X, d, W, E) then by Lemma 3.3 T is a nonexpansive mapping in $(E(X), d, W_E)$. Clearly $(E(X), d, W_E)$ is a convex metric space. Since (X, d, W, E) is strictly E-convex, then by Lemma 3.4 $(E(X), d, W_E)$ is strictly convex. Now $(E(X), d, W_E)$ is strictly convex with Property(C). Since (X, d, W, E) has E-normal structure by Lemma 3.2, $(E(X), d, W_E)$ has normal structure. By the hypothesis E(K) = K shows that K is a nonempty bounded closed convex subsets of $(E(X), d, W_E)$ with normal structure. If $\mathscr{F}$ is a commuting family of nonexpansive mappings of E(K) into E(K). By Theorem 3.5 the family has a common fixed point in E(K). Since E(K) = K this family has a common fixed point in K. This completes the proof.

**REFERENCES:**


[1] Browder F.E., Nonexpansive nonlinear operators in a Banach space, *Proc.Nat.Acad.Sci.U.S.A.* 54(1965),1041-1044.

[2] Day M.M., Fixed point theorems for compact convex sets, *Illinois J.Mat*h.5(1961),585-590.

[3] De Marr R., Common fixed–points for commuting contraction mappings, *Pacific J. Math*.13(1963),1139-1141.

[4] Kirk W.A., A fixed point theorem for mappings which do not increase distance, *Amer.Math.Monthly* 72(1965),1004-1006.

[5] Sheiba Grace J and Thangavelu P., Properties of E-Convex sets, *Tamsui Oxford Journal of Mathematical Sciences* 25(1)(2009), 1-7.

[6] Shyamala Malini S, Thangavelu P and Jeyanthi P, Fixed point theorems in E-Convex Metric Spaces, *Journal of Ultra Scientist of Physical Sciences* (to appear).

[7] Wataru Takahashi., A convexity in metric space and nonexpansive mappings, I *Kodai Math.Sem.Rep*.22(1970),142-149.

[8] Youness E.A., E-Convex sets, E-Convex functions and E-Convex programming, *J.Optim.Theory Appn.,* 102(3) (1999), 439-450.


 