International Journal of Mathematical Archive-3(7), 2012, 2827-2835
IJMA Available online through www.ijma.info ISSN 2229 – 5046

ON A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY GENERALIZED SALAGEAN OPERATOR

Saurabh Porwal[a], K. K. Dixit[b], A. L. Pathak[c] and R. Tripathi[c*]

[a] *Department of Mathematics, U.I.E.T. Campus, C.S.J.M. University, Kanpur-208024*

[b] *Department of Mathematics, Gwalior Institute of Information Technology, Gwalior-474015 (M.P.), India*

[c] *Department of Mathematics, Brahmanand College, The Mall, Kanpur(U.P.), India*

(Received on: 15-03-12; Accepted on: 18-05-12)

ABSTRACT

The purpose of the present paper is to study some results involving coefficient conditions, extreme points, distortion bounds, convolution conditions and convex combination for a new class of generalized Salagean-Type harmonic univalent functions in the open unit disc. Relevant connections of the results presented here with various known results are briefly indicated.

AMS 2010 Mathematics Subject Classification: *30C45, 30C50, 30C55.*

Keywords and Phrases: *Harmonic, Univalent functions, Convex and Starlike functions.*

1. INTRODUCTION

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that

$$|h'(z)| > |g'(z)|, z \in D.$$

Let S_H denote the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense preserving in the open unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1 . \tag{1.1}$$

Clunie and Sheil-Small [3] investigated the class S_H as well as its geometric subclasses and established some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

For $f = h + \overline{g}$ given by (1.1), we defined the modified generalized Salagean operator of f as

$$D_\lambda^m f(z) = D_\lambda^m h(z) + (-1)^m \overline{D_\lambda^m g(z)}, \quad (m \in N_0, N_0 = N \cup \{0\}, 0 \le \lambda \le 1) \tag{1.2}$$

where

$$D_\lambda^m h(z) = z + \sum_{k=2}^{\infty} \{(k-1)\lambda + 1\}^m a_k z^k$$

Corresponding author: *R. Tripathi[c*]*
[c] *Department of Mathematics, Brahmanand College, The Mall, Kanpur(U.P.), India*

and
$$D_{\lambda}^{m} g(z)=\sum_{k=1}^{\infty}\{(k-1)\lambda+1\}^{m}\, b_k z^k,$$

where D_{λ}^{m} stands for the generalized Salagean operator introduced by Al-Obaudi[8]. For $\lambda=1$ it reduces to Salagean operator introduced by Salagean [10].

Now for $0\le\gamma<1$, $\beta\ge 0$, $m\in N$, $n\in N_0$, $m>n$, $\alpha\in R, 0\le\lambda\le 1$ and $z\in U$, suppose that $RS_H(m,n;\beta;t;\gamma,\lambda)$ denote the family of harmonic functions f of the form (1.1) such that

$$\operatorname{Re}\left\{(1+\beta e^{i\alpha})\frac{D^m f(z)}{D^n f_t(z)}-\beta e^{i\alpha}\right\}\ge\gamma, \tag{1.3}$$

where $D^m f$ is defined by (1.2) and $f_t(z)=(1-t)z+\left(h(z)+\overline{g(z)}\right)t, 0\le t\le 1.$

Further, let the subclass $\overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$ consist of harmonic univalent functions $f_m=h+\overline{g_m}$ in $RS_H(m,n;\beta;t;\gamma,\lambda)$ so that h and g_m are of the form

$$h(z)=z-\sum_{k=2}^{\infty}|a_k|z^k,\ g_m(z)=(-1)^{m-1}\sum_{k=1}^{\infty}|b_k|z^k. \tag{1.4}$$

By specializing the parameters in subclass $RS_H(m,n;\beta;t;\gamma,\lambda)$ we obtain the following known subclasses studied earlier by various authors.

1. $RS_H(n+q,n;\beta;1;\gamma,1)\equiv R_H(n,\gamma,\beta,q)$ studied by Dixit et al. [4].
2. $RS_H(n+1,n;1;1;\gamma,1)\equiv RS_H(n,\gamma)$ studied by Yalcin et al. [14].
3. $RS_H(m,n;0;1;\gamma,1)\equiv S_H(m,n;\gamma)$ studied by Yalcin [13].
4. $RS_H(n+1,n;0;1;\gamma,1)\equiv H(n,\gamma)$ studied by Jahangiri et al. [6].
5. $RS_H(2,1;\beta;1;\gamma,1)\equiv HCV(\beta,\gamma)$ studied by Kim et al. [7].
6. $RS_H(1,0;\beta;t;\gamma,1)\equiv G_H(\beta,\gamma,t)$ studied by Ahuja et al. [1].
7. $RS_H(1,0;1;1;\gamma,1)\equiv G_H(\gamma)$ studied by Rosy et al. [9].
8. $\overline{RS_H}(1,0;0;1;\gamma,1)\equiv S_H^{*}(\gamma)$ studied by Jahangiri [5].
9. $RS_H(1,0;0;1;0,1)\equiv S_H^{*}$ studied by Avci and Zlotkiewicz [2], Silverman [11] and Silverman and Silvia [12].

In the present paper, results involving the coefficient bounds, extreme points, distortion bounds, convex combinations for the above classes $RS_H(m,n;\beta;t;\gamma,\lambda)$ and $\overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$ of harmonic univalent functions have been investigated.

2. MAIN RESULTS

In our first theorem, we introduce a sufficient condition for function in $RS_H(m,n;\beta;t;\gamma,\lambda)$.

Theorem 2.1: *Let $f=h+\overline{g}$ be such that h and g are given by (1.1). Furthermore, let*

$$\sum_{k=2}^{\infty}\frac{\{(k-1)\lambda+1\}^m(1+\beta)-\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma}|a_k| \\ +\sum_{k=1}^{\infty}\frac{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma}|b_k|\le 1 \tag{2.1}$$

where $m\in N, n\in N_0, m>n, 0\le\gamma<1, \beta\ge 0$ and $0\le t\le 1$, then f is sense-preserving, harmonic univalent in U and $f \in RS_H(m,n;\beta;t;\gamma,\lambda)$.

Proof: If $z_1 \neq z_2$ then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty} b_k\left(z_1^k-z_2^k\right)}{\left(z_1-z_2\right)+\sum_{k=2}^{\infty} a_k\left(z_1^k-z_2^k\right)}\right|$$

$$>1-\frac{\sum_{k=1}^{\infty}\{(k-1)\lambda+1\}|b_k|}{1-\sum_{k=2}^{\infty}\{(k-1)\lambda+1\}|a_k|}$$

$$\geq 1-\frac{\left(\sum_{k=1}^{\infty}\frac{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma}|b_k|\right)}{1-\sum_{k=2}^{\infty}\frac{\{(k-1)\lambda+1\}^m(1+\beta)-\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma}|a_k|}$$

$$\geq 0, \qquad \text{(Using (2.1))}$$

which proves the univalence.

Also, we have

$$|h'(z)| \geq 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1}$$

$$>1-\sum_{k=2}^{\infty} k|a_k|$$

$$\geq 1-\sum_{k=2}^{\infty}\frac{\{(k-1)\lambda+1\}^m(1+\beta)-\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma}|a_k|$$

$$\geq \sum_{k=1}^{\infty}\frac{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma}|b_k|$$

$$> \sum_{k=1}^{\infty}\frac{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma}|b_k||z|^{k-1}$$

$$\geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1}$$

$$\geq |g'(z)|.$$

Hence f is sense-preserving in U.

Using the fact $\text{Re}(w) \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it suffices to show that

$$|(1-\gamma)D_\lambda^n f_t(z)+(1+\beta e^{i\alpha})D_\lambda^m f(z)-\beta e^{i\alpha}D_\lambda^n f_t(z)| \\ -|(1+\gamma)D_\lambda^n f_t(z)-(1+\beta e^{i\alpha})D_\lambda^m f(z)+\beta e^{i\alpha}D_\lambda^n f_t(z)| \geq 0 \tag{2.2}$$

Substituting for $D_\lambda^m f(z)$ and $D_\lambda^n f_t(z)$ in L.H.S. of (2.2) we have

$$=|(2-\gamma)z+\sum_{k=2}^{\infty}[(1-\gamma)\{(k-1)\lambda+1\}^n t+(1+\beta e^{i\alpha})\{(k-1)\lambda+1\}^m-\beta e^{i\alpha}$$

$$\{(k-1)\lambda+1\}^n t]a_k z^k+(-1)^n\sum_{k=1}^{\infty}[(1-\gamma)\{(k-1)\lambda+1\}^n t+(-1)^{m-n}(1+\beta e^{i\alpha})$$

$$\{(k-1)\lambda+1\}^m-\beta e^{i\alpha}\{(k-1)\lambda+1\}^n t]\overline{b_k z^k}|-|\gamma z+\sum_{k=2}^{\infty}[(1+\beta e^{i\alpha})\{(k-1)\lambda+1\}^m$$

$$-(1+\gamma)\{(k-1)\lambda+1\}^n t-\beta e^{i\alpha}\{(k-1)\lambda+1\}^n t]a_k z^k+(-1)^n\sum_{k=1}^{\infty}[(-1)^{m-n}$$

$$(1+\beta e^{i\alpha})\{(k-1)\lambda+1\}^m-(1+\gamma)\{(k-1)\lambda+1\}^n t-\beta e^{i\alpha}\{(k-1)\lambda+1\}^n t]\overline{b_k z^k}|$$

$$\geq 2(1-\gamma)|z|-2\sum_{k=2}^{\infty}[\{(k-1)\lambda+1\}^m(1+\beta)-(\gamma+\beta)t\{(k-1)\lambda+1\}^n]|a_k||z|^k$$

$$-\sum_{k=1}^{\infty}|[(1-\gamma)\{(k-1)\lambda+1\}^n t+(-1)^{m-n}(1+\beta e^{i\alpha})\{(k-1)\lambda+1\}^m-\beta e^{i\alpha}\{(k-1)\lambda+1\}^n t]||b_k||z|^k$$

$$-\sum_{k=1}^{\infty}|[(-1)^{m-n}(1+\beta e^{i\alpha})\{(k-1)\lambda+1\}^m-(1+\gamma)\{(k-1)\lambda+1\}^n t-\beta e^{i\alpha}\{(k-1)\lambda+1\}^n t]||b_k||z|^k$$

$$=\begin{cases} 2(1-\gamma)|z|-2\sum_{k=2}^{\infty}[\{(k-1)\lambda+1\}^m(1+\beta)-(\gamma+\beta)t\{(k-1)\lambda+1\}^n]|a_k||z|^k \\ \quad -2\sum_{k=1}^{\infty}[\{(k-1)\lambda+1\}^m(1+\beta)+(\gamma+\beta)t\{(k-1)\lambda+1\}^n]|b_k||z|^k \\ \qquad\qquad \text{if } m-n \text{ is odd} \\ 2(1-\gamma)|z|-2\sum_{k=2}^{\infty}[\{(k-1)\lambda+1\}^m(1+\beta)-(\gamma+\beta)t\{(k-1)\lambda+1\}^n]|a_k||z|^k \\ \quad -2\sum_{k=1}^{\infty}[\{(k-1)\lambda+1\}^m(1+\beta)-(\gamma+\beta)t\{(k-1)\lambda+1\}^n]|b_k||z|^k \\ \qquad\qquad \text{if } m-n \text{ is even} \end{cases}$$

$$=2(1-\gamma)|z|\{1-\sum_{k=2}^{\infty}\frac{\{(k-1)\lambda+1\}^n(1+\beta)-(\gamma+\beta)t\{(k-1)\lambda+1\}^n}{1-\gamma}|a_k||z|^{k-1}$$

$$-\sum_{k=1}^{\infty}\frac{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}(\gamma+\beta)t\{(k-1)\lambda+1\}^n}{1-\gamma}|b_k||z|^{k-1}\}$$

$$>2(1-\gamma)\{1-(\sum_{k=2}^{\infty}\frac{\{(k-1)\lambda+1\}^m(1+\beta)-(\gamma+\beta)t\{(k-1)\lambda+1\}^n}{1-\gamma}|a_k|$$

$$-\sum_{k=1}^{\infty}\frac{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}(\gamma+\beta)t\{(k-1)\lambda+1\}^n}{1-\gamma}|b_k|)\}.$$

The last expression in non-negative by (2.1) and so that proof is complete.

The harmonic univalent functions

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\gamma}{\{(k-1)\lambda+1\}^m(1+\beta)-(\gamma+\beta)t\{(k-1)\lambda+1\}^n}x_kz^k$$
$$+\sum_{k=1}^{\infty}\frac{1-\gamma}{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}(\gamma+\beta)t\{(k-1)\lambda+1\}^n}\overline{y_kz^k}. \tag{2.3}$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (2.1) is sharp.

This completes the proof of Theorem 2.1.

In the following theorem, it is proved that the condition (2.1) is also necessary for function $f_m=h+\overline{g_m}$, where h and g_m are of the form (1.4).

Theorem 2.2: *Let* $f_m=h+\overline{g_m}$ *be given by (1.4). Then* $f_m\in\overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$ *if and only if*

$$\sum_{k=1}^{\infty}(\{(k-1)\lambda+1\}^m(1+\beta)-\{(k-1)\lambda+1\}^n t(\gamma+\beta)|a_k|+\{(k-1)\lambda+1\}^m$$
$$(1+\beta)-(-1)^{m-n}\{(k-1)\lambda+1\}^n t(\gamma+\beta)|b_k|)\le 2(1-\gamma). \tag{2.4}$$

Proof: Since $RS_H(m,n;\beta;t;\gamma,\lambda)\subseteq\overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_m of the form (1.4), we notice that condition

$$Re\left[(1+\beta e^{i\alpha})\frac{D_\lambda^m f(z)}{D_\lambda^n f_t(z)}-\beta e^{i\alpha}\right]\ge\gamma$$

is equivalent to

$$Re\left\{\frac{(1-\gamma)z-\sum_{k=2}^{\infty}[\{(k-1)\lambda+1\}^m(1+\beta e^{i\alpha})-(\beta e^{i\alpha}+\gamma)\{(k-1)\lambda+1\}^n t]|a_k|z^k+(-1)^{2m-1}\sum_{k=1}^{\infty}[\{(k-1)\lambda+1\}^m(1+\beta e^{i\alpha})-(-1)^{m-n}(\beta e^{i\alpha}+\gamma)\{(k-1)\lambda+1\}^n t]|b_k|\overline{z}^k}{z-\sum_{k=2}^{\infty}\{(k-1)\lambda+1\}^n t|a_k|z^k+(-1)^{m+n-1}\sum_{k=1}^{\infty}\{(k-1)\lambda+1\}^n t|b_k|\overline{z}^k}\right\}\ge 0. \tag{2.5}$$

The above required condition (2.5) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\le z=r<1$, we must have

$$Re\left\{\frac{(1-\gamma)-\sum_{k=2}^{\infty}[\{(k-1)\lambda+1\}^m-\gamma t\{(k-1)\lambda+1\}^n]|a_k|r^{k-1} -\sum_{k=1}^{\infty}[\{(k-1)\lambda+1\}^m-(-1)^{m-n}\gamma t\{(k-1)\lambda+1\}^n]|b_k|r^{k-1}}{1-\sum_{k=2}^{\infty}\{(k-1)\lambda+1\}^n t|a_k|r^{k-1}-(-1)^{m-n}\sum_{k=1}^{\infty}\{(k-1)\lambda+1\}^n t|b_k|r^{k-1}}\right.$$

$$\left.-e^{i\alpha}\frac{\sum_{k=2}^{\infty}\beta(\{(k-1)\lambda+1\}^m-t\{(k-1)\lambda+1\}^n)|a_k|r^{k-1} -\sum_{k=1}^{\infty}\beta(\{(k-1)\lambda+1\}^m-(-1)^{m-n}t\{(k-1)\lambda+1\}^n)|b_k|r^{k-1}}{1-\sum_{k=2}^{\infty}\{(k-1)\lambda+1\}^n t|a_k|r^{k-1}-(-1)^{m-n}-\sum_{k=1}^{\infty}\{(k-1)\lambda+1\}^n t|b_k|r^{k-1}}\right\}\geq 0.$$

Since $Re(-e^{i\alpha})\geq -|e^{i\alpha}|=-1$, the above inequality reduces to

$$\frac{(1-\gamma)-\sum_{k=2}^{\infty}[\{(k-1)\lambda+1\}^m(1+\beta)-(\beta+\gamma)\{(k-1)\lambda+1\}^n t]|a_k|r^{k-1} -\sum_{k=1}^{\infty}[\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}(\beta+\gamma)\{(k-1)\lambda+1\}^n t]|b_k|r^{k-1}}{1-\sum_{k=2}^{\infty}\{(k-1)\lambda+1\}^n t|a_k|r^{k-1}-(-1)^{m-n}\sum_{k=1}^{\infty}\{(k-1)\lambda+1\}^n t|b_k|r^{k-1}}\geq 0. \quad (2.6)$$

If the condition (2.4) does not hold, then the numerator in (2.6) is negative for r sufficiently close to 1. Hence there exists a $z_0=r_0$ in (0,1) for which the quotient in (2.6) is negative. This contradicts the required condition for $f_m\in\overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$ and so the proof is complete.

We employ the techniques of Dixit et al. [1.4] in the proof of Theorem 2.3 and 2.4.

Theorem 2.3: *Let* f_m *be given by (1.4). Then* $f_m\in\overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$ *if and only if*

$$f_m(z)=\sum_{k=1}^{\infty}(x_k h_k(z)+y_k g_{mk}(z)),\ \text{where}\ h_1(z)=z,$$

$$h_k(z)=z-\frac{1-\gamma}{\{(k-1)\lambda+1\}^m(1+\beta)-(\gamma+\beta)t\{(k-1)\lambda+1\}^n}z^k,\ (k=2,3,4...),$$

$$g_{mk}(z)=z+(-1)^{m-1}\frac{1-\gamma}{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}(\gamma+\beta)t\{(k-1)\lambda+1\}^n}\overline{z}^k,\ (k=1,2,3...),$$

$x_k \ge 0$, $y_k \ge 0$, $\sum_{k=1}^{\infty}(x_k + y_k) = 1$. In particular the extreme points of $\overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$ are $\{h_k\}$ and $\{g_{mk}\}$.

Theorem 2.4: *Let* $f_m \in \overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$. *Then for* $|z| = r < 1$ *we have*

$$|f_m(z)| \le (1+|b_1|)r + \frac{1}{(\lambda+1)^n}\left(\frac{1-\gamma}{(\lambda+1)^{m-n}(1+\beta)-t(\gamma+\beta)} - \frac{(1+\beta)-(-1)^{m-n}t(\gamma+\beta)}{(\lambda+1)^{m-n}(1+\beta)-t(\gamma+\beta)}|b_1|\right)r^2$$

and

$$|f_m(z)| \ge (1+|b_1|)r - \frac{1}{(\lambda+1)^n}\left(\frac{1-\gamma}{(\lambda+1)^{m-n}(1+\beta)-t(\gamma+\beta)} - \frac{(1+\beta)-(-1)^{m-n}t(\gamma+\beta)}{(\lambda+1)^{m-n}(1+\beta)-t(\gamma+\beta)}|b_1|\right)r^2.$$

The following covering result follows from the left hand inequality in Theorem 2.4.

Corollary 2.5: *Let* f_m *of the form (1.4) be so that* $f_m \in \overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$. *Then*

$$\{w : |w| < \frac{(\lambda+1)^m(1+\beta)-(\lambda+1)^n(\gamma+\beta)t-1+\gamma}{(\lambda+1)^m(1+\beta)-(\lambda+1)^n(\gamma+\beta)t} - \frac{((\lambda+1)^m-1)(1+\beta)-t(\gamma+\beta)((\lambda+1)^n-(-1)^{m-n})}{(\lambda+1)^m(1+\beta)-((\lambda+1))^n(\gamma+\beta)t}|b_1|\}.$$

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f_m(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + (-1)^{m-1}\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$

and

$$F_m(z) = z - \sum_{k=2}^{\infty}|A_k|z^k + (-1)^{m-1}\sum_{k=1}^{\infty}|B_k|\overline{z}^k$$

we define the convolution of two harmonic functions f and F as

$$(f_m * F_m)(z) = f_m(z) * F_m(z) = z - \sum_{k=2}^{\infty}|a_k A_k|z^k + (-1)^{m-1}\sum_{k=1}^{\infty}|b_k B_k|\overline{z}^k. \qquad (2.7)$$

Using the definition, we show that the class $\overline{S_H}(m,n;\alpha;\lambda)$ is closed under convolution.

Theorem 2.6: *For* $0 \le \gamma_1 \le \gamma_2 < 1$ *let* $f_m \in \overline{RS_H}(m,n;\beta;t;\gamma_1,\lambda)$ *and* $F_m \in \overline{RS_H}(m,n;\beta;t;\gamma_2,\lambda)$. *Then* $f_m * F_m \in \overline{RS_H}(m,n;\beta;t;\gamma_2,\lambda) \subseteq \overline{RS_H}(m,n;\beta;t;\gamma_1,\lambda)$.

Proof: Let $f_m(z) = z - \sum_{k=2}^{\infty}a_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty}b_k\overline{z}^k$ be in $\overline{RS_H}(m,n;\beta;t;\gamma_1,\lambda)$ and

$$F_m(z) = z - \sum_{k=2}^{\infty}A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty}B_k z^k$$

be in $\overline{RS_H}(m,n;\beta;t;\gamma_2,\lambda)$. Then the convolution $f_m * F_m$ is given by (2.7). We wish to show that the coefficients of $f_m * F_m$ satisfy the required condition given in Theorem 2.2. For $F_m \in \overline{RS_H}(m,n;\beta;t;\gamma_2,\lambda)$ we note that

$A_k \leq 1$ and $B_k \leq 1$. Now, for the convolution function $f_m * F_m$ we get

$$\sum_{k=2}^{\infty} \frac{\{(k-1)\lambda+1\}^m(1+\beta)-\{(k-1)\lambda+1\}^n t(\gamma_2+\beta)}{1-\gamma_2} |a_k A_k|$$

$$+\sum_{k=1}^{\infty} \frac{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}\{(k-1)\lambda+1\}^n t(\gamma_2+\beta)}{1-\gamma_2} |b_k B_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{\{(k-1)\lambda+1\}^m(1+\beta)-\{(k-1)\lambda+1\}^n t(\gamma_1+\beta)}{1-\gamma_1} |a_k|$$

$$+\sum_{k=1}^{\infty} \frac{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}\{(k-1)\lambda+1\}^n t(\gamma_1+\beta)}{1-\gamma_1} |b_k|$$

$$\leq 1. \qquad (\text{Since } f_m \in \overline{RS_H}(m,n;\beta;t;\gamma_1,\lambda))$$

Therefore $f_m * F_m \in \overline{RS_H}(m,n;\beta;t;\gamma_2,\lambda) \subseteq \overline{RS_H}(m,n;\beta;t;\gamma_1,\lambda)$ for $0 \leq \beta \leq \alpha < 1$.

Now we show that $\overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$ is closed under convex combinations.

Theorem 2.7: *The class* $\overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$ *is closed under convex combination.*

Proof: For $i = 1,2,3...$ let $f_{m_i}(z) \in \overline{RS_H}(m,n;\beta;t;\gamma,\lambda),$ where $f_{m_i}(z)$ is given by $f_{m_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_{k_i}\overline{z}^k$.

Then by Theorem 2.2

$$\sum_{k=1}^{\infty} \frac{\{(k-1)\lambda+1\}^m(1+\beta)-\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma} |a_{k_i}|$$

$$+\frac{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma} |b_{k_i}| \leq 2.$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) z^k + (-1)^{m-1}\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{k_i}\right)\overline{z}^k.$$

Then by 2.2,

$$\sum_{k=1}^{\infty}[\frac{\{(k-1)\lambda+1\}^m(1+\beta)-\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma} \sum_{i=1}^{\infty} |t_i a_{k_i}|$$

$$+\frac{\{(k-1)\lambda+1\}^m(1+\beta)-(-1)^{m-n}\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma} \sum_{i=1}^{\infty} |t_i b_{k_i}|] \leq 2$$

$$= \sum_{i=1}^{\infty} t_i \{\sum_{k=1}^{\infty} \frac{\{(k-1)\lambda+1\}^m(1+\beta)-\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma} |a_{k_i}|$$

$$+\frac{\{(k-1)\lambda+1\}^n(1+\beta)-(-1)^{m-n}\{(k-1)\lambda+1\}^n t(\gamma+\beta)}{1-\gamma} |b_{k_i}|$$

$$\leq 2\sum_{i=1}^{\infty} t_i = 2.$$

This is the condition required by Theorem 2.2 and so $\sum_{i=1}^{\infty} t_i f_{m_i}(z) \in \overline{RS_H}(m,n;\beta;t;\gamma,\lambda)$.

[1]ACKNOWLEDGEMENT

The author is thankful to University Grants Commission, New Delhi, Govt. of India for financial support under grant serial no. F.No. 8-2 (223)/2011 (MRP/NRCB).

REFERENCES

[1] Ahuja, O.P., Aghalary, R. and Joshi, S.B. (2005). ``Harmonic univalent functions associate with k-Uniformly starlike functions", J. Math. Res. Sci, 9(1), 9-17.

[2] Avci, Y. and Zlotkiewicz, E. (1990). ``On harmonic univalent mappings". Ann. Univ. Mariae Curie-Sklodowska, Sect.A 44, 1-7.

[3] Clunie, J. and Sheil-Small, T. (1984). ``Harmonic univalent functions", Ann. Acad. Sci. Fenn. Ser. AI Math 9, 3-25.

[4] Dixit, K.K., Pathak, A.L., Porwal, S. and Agarwal, R. (2009), ``A new subclass of harmonic univalent functions defined by Salagean operator", Int. J. Contemp. Math. Sci. 4(8), 371-383.

[5] Jahangiri, J.M., (1999). ``Harmonic functions starlike in the unit disc", J. Math. Anal. Appl. 235, 470-477.

[6] Jahangiri, J.M., Murugusundaramoorthy, G. and Vijay, K. (2002), ``Salagean-type harmonic univalent functions", South J. Pure Appl. Math. 2, 77-82.

[7] Kim, Y.C., Jahangiri, J.M. and Choi, J.H. (2002), ``Certain convex harmonic functions", Int. J. Math. Math. Sci., 29 (8), 459-465.

[8] Oboudi, Al. (2004), On Univalent function defined by Generalized Salagean operator, International Journal of Mathematics and Mathematical Sciences, 27, 1429-1436.

[9] Rosy, T, Stephen, B.A., Subramanian, K.G. and Jahangiri, J.M. (2001), ``Goodman-Ronning-type harmonic univalent functions", Kyungpook Math. J. 41 (1), 45-54.

[10] Salagean, G.S. (1983) ``Subclasses of univalent functions", Complex Analysis Fifth Romanian Finish Seminar, Bucharest 1, 362-372.

[11] Silverman, H. (1998) ``Harmonic univalent functions with negative coefficients" J. Math. Anal. Appl. 220, 283-289.

[12] Silverman, H. and Silvia, E.M. (1999), ``Subclass of harmonic univalent functions", New Zealand J. Math. 28, 275-284.

[13] Yalcin, S. (2005), ``A new class of Salagean-type harmonic univalent functions", Appl. Maths. Letters 18, 191-198.

[14] Yalcin, S., Öztürk and Yamankaradeniz, M. (2007), ``On the subclass of Salagean-type harmonic univalent functions", J. Inequl. Pure Appl. Maths. 8(2), Art 54, 1-17.

Source of support: University Grants Commission, New Delhi, Govt. of India, Conflict of interest: None Declared

