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ON SET-VALUED SOLUTION OF THE FEIGENBAUM TYPE FUNCTIONAL EQUATION*

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ABSTRACT

Extended Feigenbaum functional equation is considered for multivalued functions in Banach space. The existence of its solution is obtained by Banach's fixed point theorem; Furthermore continuous dependence of the solution upon the given function is obtained.

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1. INTRODUCTION:

The second kind of Feigenbaum functional equation

$$\begin{cases} f(x)=\dfrac{1}{\lambda}f(f(\lambda x)), 0<\lambda<1; \\ f(0)=1; \\ 0\le f(x)\le 1, x\in[0,1] \end{cases} \tag{1.1}$$

is considered by many authors [1-6], where they discussed its single peak continuous solutions in the interval [0,1] with the initial value condition, also considered the precise solution in some special functional space .Of course, from the view of dynamics system , it is a very important field. Considering it contains iteration,

one is also interesting in studying its continuous solution in high dimensional space and abstract space such as Banach space, Especially to study the iterative equation (1.1) for set-valued functions. The existence and uniqueness of upper semicontinuous set-valued solutions of iterative equation

$$\lambda_1 f(x)+\lambda_2 f^2(x)=g(x) \tag{1.2}$$

are studied in [9] for a class of set-valued functions on a compact real interval $[a,b]$ and the end-points a and b are

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fixed, i.e., $f(a)=a$ and $f(b)=b$. In comparison with those known results in single-valued sense, more difficulties are encountered in [9] ,because the properties of set-valued functions are different from that of single-valued functions ,such as monotionicity, differentiability, invertibility and so on.

In this paper, by constructing a structure operator and using Banach's fixed point theorem, an extended Feifenbaum type functional equation

$$f(x)=\frac{1}{\lambda}f(f(\lambda x))+\lambda g(x), \tag{1.3}$$

is studied in a Banach space $(X,\|.\|)$, where f(x) and g(x) are multi-valued functions on Banach space $(X,\|.\|)$.The existence and uniqueness of continuous set-valued solutions to equation (1.3) are obtained without monotonicity. Furthermore, conditions dependence of the solution on the given function is also presented.

2. PRELIMINARIES:

Let $(X,\|.\|)$ be a Banach space, t a real number and A,B be subsets of X. We define

$$A+B:=\{a+b:a\in A,b\in B\},$$

$$tA:=\{ta:a\in A\}.$$

Let $D\subset X$ be a given nonempty convex compact subset and $cc(D)$ denote the family of all nonempty convex compact subsets of D. The family endowed with the Hausdorff distance defined by

$$h(A,B)=\max\{\sup\{d(a,B):a\in A\},\sup\{d(b,A):b\in B\}\}, \tag{2.1}$$

Where

$$d(a,B)=\inf\{\|a-b\|:b\in B\},$$

is a complete metric space (cf.e.g[7],Theorem 10.1.6) ,especially,

$$h(x,y)=\|x-y\|,\forall x,y\in D.$$

Let $\mathrm{F}(D)$ be a family of all set-valued functions $f:D\to cc(D)$ and for some constant M, let $\phi(D,M)$ be its subfamily defined by

$$\phi(D,M)=\{f\in \mathrm{F}(D):h(f(x),f(y))\le Mh(x,y) \text{ for any } x,y\in D\},$$

i.e. $\phi(D,M)$ is all the Lipschitz multi-valued function and M is a Lipschitz constant.

We endow $\phi(D,M)$ with the metric

$$H(f_1,f_2)=\sup\{h(f_1(x),f_2(x)):x\in D\},\ \forall f_1,f_2\in\phi(D,M). \tag{2.2}$$

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Lemma: 2.1 Let A, B, $C \in cc(D)$.Then

$$h(A-C, B-C) \le h(A, B)$$

and

$$h(A+C, B+C) \le h(A, B)$$

Proof: For any $r \in (A-C)$ and $s \in (B-C)$, there exist $a \in A, b \in B$,and $c_1, c_2 \in C$ such that $r = a - c_1$, $s = b - c_2$. Since

$$\|r-s\| = \|(a-c_1)-(b-c_2)\|$$

$$\le \|a-b\| + \|c_1 - c_2\|$$

So we have

$$\begin{aligned}\inf_{s\in B-C}\|r-s\| &\le \inf_{s\in B-C}(\|a-b\| + \|c_1-c_2\|)\\ &= \inf_{b-c_2\in B-C}\|a-b\| + \inf_{b-c_2\in B-C}\|c_1-c_2\|\\ &= \inf_{b\in B}\|a-b\| + \inf_{c_2\in C}\|c_1-c_2\|\\ &= \inf_{b\in B}\|a-b\|.\end{aligned}$$

By (2.1) we have

$$\begin{aligned}\sup\{d(r, B-C): r\in A-C\} &= \sup_{r\in A-C}\{\inf_{s\in A-C}\|r-s\|\}\\ &\le \sup_{a-c_1\in A-C}\{\inf_{b\in B}\|a-b\|\}\\ &= \sup_{a\in A}\{\inf_{b\in B}\|a-b\|\}\\ &= h(A, B).\end{aligned}$$

Similarly , we can prove that

$$\sup\{d(s, A-C): s\in B-C\} = \sup_{s\in B-C}\{\inf_{r\in A-C}\|r-s\|\} \le h(A, B).$$

Hence, by (2.1), we have

$$h(A-C, B-C) \le h(A, B)$$

Similarly, we can get $h(A+C, B+C) \le h(A, B)$.

Lemma: 2.2. The metric space $(\phi(D, M), H)$ is complete.

Proof: Let f_n be a Cauchy sequence in $\phi(D, M)$.Then for every fixed $x \in D$, $f_n(x)$ is a Cauchy sequence in

$(cc(D),h)$, since $(cc(D),h)$ is completed.

(1) We will show that $f \in \phi(D,M)$

Obviously, we only prove that for any $x, y \in D$

$$h(f(x), f(y)) \leq Mh(x, y).$$

In fact, for any $\varepsilon > 0$,there exists a $n_0 \in N$ such that $h(f_n(x), f(x)) \leq \varepsilon$ and $h(f_n(y), f(y)) \leq \varepsilon$ for all $n > n_0$ .Hence

$$h(f(x), f(y)) \leq h(f_n(x), f(x)) + h(f_n(x), f_n(y)) + h(f_n(y), f(y))$$

$$\leq 2\varepsilon + Mh(x, y).$$

Form that the ε is arbitrary, we get that

$$h(f(x), f(y)) \leq Mh(x, y)$$

(2) We will prove that $f_n \to f$ in the sense of the metric H

In fact, for arbitrary $\varepsilon > 0$, since f_n is a Cauchy sequence ,there exists a $n_0 \in N$ such that $H(f_n(x), f_k(x)) \leq \varepsilon$ for all $n, k \geq n_0$ .Hence

$$h(f_n(x), f_k(x)) \leq \varepsilon, \qquad \forall x \in D$$

Let $k \to \infty$, then we have

$$h(f_n(x), f(x)) \leq \varepsilon, \qquad \forall x \in D$$

Which means that

$$H(f_n, f) \leq \varepsilon$$

Form (1) and (2) , we obtain that $(\phi(D,M), H)$ is completed metric space.

Lemma: 2.3 Let $f \in \phi(D,M)$. Then for any $K \in cc(D)$,

$$f(K) \in cc(D).$$

Where $f(K) = \bigcup_{x \in K} f(x)$.

Proof: See, for instance, Lemma 10.4.2 in [7].

Lemma: 2.4 Let $A, B \in cc(D)$ and $f \in \phi(D,M)$ .Then

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$$h(f(A), f(B)) \le Mh(A, B).$$

Proof: See , for instance ,lemma 10.4.3 in [7].

3. EXISTENCE OF SOLUTIONS:

Theorem: 1 Suppose that $g \in \phi(D, m)$ and there exists a constant M satisfies that $m \le \frac{1}{\lambda}(M - M^2)$ and $\lambda > 1 + M$ .Then the equation (1.3) has a solution $f \in \phi(D, M)$.

Proof Define the mapping $\tau : \phi(D, M) \to \mathrm{F}(D)$ by

$$\tau(f)x = \frac{1}{\lambda} f(f(\lambda x)) + \lambda g(x), \forall f \in \phi(D, M), \quad x \in D \tag{3.1}$$

By Lemma 2.3, we know $\tau(f) \in \mathrm{F}(D)$, and we will prove the theorem in following two steps.

(1) We will show that $\tau : \phi(D, M) \to \phi(D, M)$.

At first, we will show that

$$h(\tau(f)x, \tau(f)y) \le Mh(x, y) \quad , \quad \forall x, y \in D \tag{3.2}$$

In fact, for any $x, y \in D$, by Lemma 2.1 and Lemma 2.4 we have

$$\begin{aligned}
h(\tau(f)x, \tau(f)y) &= h(\frac{1}{\lambda} f(f(\lambda x)) + \lambda g(x), \frac{1}{\lambda} f(f(\lambda y)) + \lambda g(y)) \\
&\le h(\frac{1}{\lambda} f(f(\lambda x)) + \lambda g(x), \frac{1}{\lambda} f(f(\lambda x)) + \lambda g(y)) \\
&+ h(\frac{1}{\lambda} f(f(\lambda x)) + \lambda g(y), \frac{1}{\lambda} f(f(\lambda y)) + \lambda g(y)) \\
&\le \frac{1}{\lambda} h(f(f(\lambda x)), f(f(\lambda y)) + \lambda h(g(x), g(y)) \\
&\le \frac{1}{\lambda} M^2 \lambda h(x, y) + \lambda m h(x, y) \\
&= (M^2 + \lambda m) h(x, y).
\end{aligned}$$

Form $m \le \frac{1}{\lambda}(M - M^2)$ .it can be deduced that $M^2 + \lambda m \le M$ .So we obtain

$\tau f \in \phi(D, M)$.

(2) We will prove that τ is contraction mapping.

For any $f_1, f_2 \in \phi(D, M)$, $x \in D$,by Lemma 2.1,2.3 and 2.4.we have

$$h(\tau(f_1)x,\tau(f_2)x)=h(\frac{1}{\lambda}f_1(f_1(\lambda x))+\lambda g(x),\frac{1}{\lambda}f_2(f_2(\lambda x))+\lambda g(x))$$

$$\le\frac{1}{\lambda}h(f_1^2(\lambda x),f_2^2(\lambda x))$$

$$\le\frac{1}{\lambda}(h(f_1^2(\lambda x),f_1\circ f_2(\lambda x))+h(f_1\circ f_2(\lambda x),f_2^2(\lambda x)))$$

$$\le\frac{1}{\lambda}(Mh(f_1(\lambda x),f_2(\lambda x))+h(f_1\circ f_2(\lambda x),f_2^2(\lambda x)))$$

$$\le\frac{1}{\lambda}(M+1)H(f_1,f_2).$$

By (2.2) and that $x\in D$ is arbitrary, we get

$$H(\tau(f_1),\tau(f_2))\le\frac{1}{\lambda}(M+1)H(f_1,f_2).$$

From $\lambda>1+M$,it is follows that $\frac{1}{\lambda}(1+M)<1$.Therefore, by Lemma2.2 and Banach's fixed point theorem, τ has a fixed point $f\in\phi(D,M)$,i.e.

$$\tau(f)x=f(x),\forall x\in D$$

Hence we obtain

$$\frac{1}{\lambda}f(f(\lambda x))+\lambda g(x)=f(x),\forall x\in D$$

This completes the proof.

4. CONTINUOUS DEPENDENCE:

Theorem: 2 Under condition of Theorem 1,the solution of equation (1.3) depends upon g continuously in the space $(\phi(D,M),H)$.

Proof: For arbitrary $g_1,g_2\in\phi(D,M)$,let f_1,f_2 be the corresponding solutions of equation (1.3),that is,

$$f_1(x)=\frac{1}{\lambda}f_1(f_1(\lambda x))+\lambda g_1(x),\ f_2(x)=\frac{1}{\lambda}f_2(f_2(\lambda x))+\lambda g_2(x).$$

Hence by Lemma 2.1 and Lemma 2.4, for arbitrary $x\in D$ we get

$$h(f_1(x),f_2(x))=h(\frac{1}{\lambda}f_1(f_1(\lambda x))+\lambda g_1(x),\frac{1}{\lambda}f_2(f_2(\lambda x))+\lambda g_2(x)).$$

$$\le h(\frac{1}{\lambda}f_1(f_1(\lambda x))+\lambda g_1(x),\frac{1}{\lambda}f_1(f_1(\lambda x))+\lambda g_2(x))+h(\frac{1}{\lambda}f_1(f_1(\lambda x))$$

$$+\lambda g_2(x),\frac{1}{\lambda}f_2(f_2(\lambda x))+\lambda g_2(x)).$$

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$$\le \frac{1}{\lambda} h(f_1^2(\lambda x), f_2^2(\lambda x)) + \lambda h(g_1(x), g_2(x))$$
$$\le \frac{1}{\lambda} h(f_1^2(\lambda x), f_1 \circ f_2(\lambda x)) + \frac{1}{\lambda} h(f_1 \circ f_2(\lambda x), f_2^2(\lambda x)) + \lambda h(g_1(x), g_2(x))$$
$$\le \frac{1}{\lambda} M h(f_1(x), f_2(x)) + \frac{1}{\lambda} h(f_1 \circ f_2(x), f_2^2(x)) + \lambda h(g_1(x), g_2(x)).$$

By (2.2) and the assumption of the Theorem 1, it follows that

$$H(f_1, f_1) \le \frac{\lambda^2}{\lambda - M - 1} H(g_1, g_2).$$

Which implies the conditions dependence of the solution f of (1.3) upon

the given g.

5. EXAMPLE:

Let $I = [0,1]$ and the multifunction

$$g(x) = [\frac{x^3}{36}, \frac{x^2}{36}], \forall x \in I.$$

For any $y \in I$ and $y \neq x$.

$$h(g(x), g(y)) = \max\left\{\left|\frac{x^3}{36} - \frac{y^3}{36}\right|, \left|\frac{x^2}{36} - \frac{y^2}{36}\right|\right\}$$
$$= \frac{1}{36} \max\left\{\left|(x-y)(x^2+xy+y^2)\right|, \left|(x-y)(x+y)\right|\right\}$$
$$\le \frac{1}{12}|x-y|$$
$$= \frac{1}{12} h(x, y).$$

So $g \in \phi(I, \frac{1}{12})$. By the Theorem 1 the equation

$$f(x) = \frac{2}{3} f(f(\frac{3}{2}x)) + \frac{3}{2} g(x)$$

has a solution in $\Phi(I, \frac{1}{4}), because \frac{1}{12} < \frac{2}{3}(\frac{1}{4} - (\frac{1}{4})^2) = \frac{1}{8} and \frac{3}{2} > 1 + \frac{1}{4}.$

REFERENCE:

[1] Lanford O E. A computer assisted proof of the Feigenbaum conjectures. Bull Amer Math Soc,6:427-434(1987)
[2] Eokmann J P. Wittwer P. A computer assisted proof of the Feigenbaum conjectures. J Stat phys. 46:455-477(1987)
[3] Epstein H. New proofs of the existence of the Feigenbaum function. Comm.,Math phys.106:395-426(1986)
[4] Thompson C J.McGuire J B. Asymptotic and essentially singular solution of the Feigenbaum equation. J Stat phys, 27:183-200(1982)

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[5] Mestel B D.Osbaldestin A H.Tstgvintsev A V. Continued fractions and solutions of the Feigenbaum-Cvitanovie equation. C R Acad Sci Paris.334:683-688 (2002)

[6] McCarthy P.J. The general exact bijective continuous solution of Feigenbaum's functional equation .Comm Math Phys.91:431-443(1983)

[7] Vasile I. Istratescu. Fixed point Theory [M].Holland, Boston ,London: D.Reidel publishing Companny .1981

[8] J.H.Mai and X.H.Liu.Existence,uniqueness and stability of c^m solution of iterative functional equations[J].Sci.in China A43 (2000.9:897-913).

[9] K.Nikodem and Weinian Zhang. On a multivalued iterative equation[J]. Publ. Math. Debrecen 64/3-4(2004),427-435.

[10] Xiaopei Li. A Class of Iterative Equation on a Banach Space[J]. Journal of Sichuan University (N.SCI.E).2004,41(3):505-510.

[11] E.Kleim and A.C.Thompson. Theory of Correspondence[M].John Willey & Sons, New York ,Chichester, Brisbane, toronto , Singapore ,1984.

[12] S.Hu and N.S. Papageorgiou. Handbook of Multivalued Analysis[M]. Dordrecht, Boston , London :Kluwer Academic Publishers. 1997.

[13] W.Jarczyk. On an equation of linear iteration[J]. Aequations Math. 1996,51:303- 310.

[14] M.Kulczyk, J.Tabor. Iterative functional equations in the class of Lipschitz func- tions[J]. Aequationes Math. 2002,64:24-33.

[15] J. P. AUBIN and H. FRANKOWSKA. Set-valued Analysis[M].Birkhauser, Bos- ton.Basel, Berlin. 1990

[16] L. GORNIEWICZ. Topological Fixed Point Theory of Multivalued Mappings[M] Kluwer Academic Pulishers , Dordrecht, Boston, London .1999.

[17] K. NIKODEM. On Jensen's functional equation for set-valued functions[J]. Radovi Mat. 1987, 3:23-33.

[18] K. NIKODEM. Set-valued solutions of the Pexider functional equation[J].Funk- cial. Ekvac. 1988, 31:227-231.

[19] T. Powierza. Set-valued iterative square foots of bijections . Bull. Acad. Polon.Sci. Ser. Sci. Math.1999, 47:377-383.

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