International Journal of Mathematical Archive-3(7), 2012, 2799-2808
Available online through www.ijma.info ISSN 2229 – 5046

GROWTH OF DIFFERENTIAL MONOMIALS AND DIFFERENTIAL POLYNOMIALS GENERATED BY ENTIRE OR MEROMORPHIC FUNCTIONS IN THE LIGHT OF ZERO ORDER

Sanjib Kumar Datta[1*], Tanmay Biswas[2] and Dilip Chandra Pramanik[3]

[1]Department of Mathematics, University of Kalyani, Kalyani, Dist-Nadia, PIN- 741235, WB, India
[2]Rajbari, Rabindrapalli, R. N. Tagore Road, P. O. Krishnagar, Dist-Nadia, PIN- 741101, WB, India
[3]Malda Railway (H.S.), P.O. - Jhaljhalia, Dist.-Malda, PIN- 732102, WB, India

(Received on: 02-06-12; Revised & Accepted on: 20-06-12)

ABSTRACT

The comparative growth rates of composite entire or meromorphic functions and differential monomials, differential polynomials generated by one of the factors have been investigated in this paper.

AMS Subject Classification (2010): *30D35, 30D30.*

Keywords and phrases: *Entire and meromorphic function, differential monomial, differential polynomial, composition, growth, zero order.*

1. INTRODUCTION, DEFINITIONS AND NOTATIONS

For any two transcendental entire functions f and g defined in the open complex plane $\mathbb{C}$, Clunie [4] proved that

$$\lim_{r\to\infty}\frac{T(r,fog)}{T(r,f)}=\infty \qquad and \qquad \lim_{r\to\infty}\frac{T(r,fog)}{T(r,g)}=\infty.$$

Singh [13] proved some comparative growth properties of $logT(r,fog)$ and $T(r,f)$. He also raised the problem of investigating the comparative growth of $logT(r,fog)$ and $T(r,f)$ which he was unable to solve. However, some results on the comparative growth of $logT(r,fog))$ and $T(r,g)$ are proved in [8]

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_{0j},n_{1j},\ldots.n_{kj}\,(k\geq 1)$ be non-negative integers such that for each $j,\sum_{i=0}^{k}n_{ij}\geq 1$. We call $M_j[f]=A_j(f)^{n_{0j}}\left(f^{(1)}\right)^{n_{1j}}\ldots\left(f^{(k)}\right)^{n_{kj}}$ where $T\left(r,A_j\right)=S(r,f)$ to be a differential monomial generated by f. The number $\gamma_{M_i}=\sum_{i=0}^{k}n_{ij}$ and $\Gamma_{M_i}=\sum_{i=0}^{k}(i+1)n_{ij}$ are called respectively the degree and weight of $M_j[f]$ {[6],[12]}. The expression $P[f]=\sum_{j=1}^{s}M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P=\max_{1\leq j\leq s}\gamma_{M_j}$ and $\Gamma_P=\max_{1\leq j\leq s}\Gamma_{M_j}$ are called respectively the degree and weight of $P[f]$ {[6],[12]}. Also we call the numbers $\underline{\gamma}_P=\min_{1\leq j\leq s}\gamma_{M_j}$ and k (the order of the highest derivative of (f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma}_p=\gamma_P, P[f]$ is called a homogeneous differential polynomial. In the paper we further investigate the question of Singh [13] mentioned earlier and prove some new results relating to the comparative growths of composite entire or meromorphic functions and differential monomials, differential polynomials generated by one of the factors. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [16] and [7]. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e., for which $n_{0j}=0$ for $j=1,2,\ldots s$. We consider only those $P[f]$, $P_0[f]$ singularities of whose individual terms do not cancel each other. We also denote by $M[f]$ a differential monomial generated by transcendental meromorphic function f.

Corresponding author: *Sanjib Kumar Datta[1*]*
[1]Department of Mathematics, University of Kalyani, Kalyani, Dist-Nadia, PIN- 741235, WB, India

The following definitions are well known.

Definition 1: The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r},$$

Where $log^{[k]}x = log(log^{[k-1]}x)$ for $k = 1,2,3,\ldots$ and $log^{[0]}x = x$.

If $\rho_f < \infty$ then f is of finite order. Also $\rho_f = 0$ means that f is of order zero. In this connection Datta and Biswas [5] gave the following definition.

Definition 2: [5] Let f be a meromorphic function of order zero. Then the quantities ρ_f^{**} and λ_f^{**} of f are defined by:

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{T(r,f)}{logr} \quad and \quad \lambda_f^{**} = \liminf_{r\to\infty} \frac{T(r,f)}{logr}.$$

If f is an entire function then clearly

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{logM(r,f)}{logr} \quad and \quad \lambda_f^{**} = \liminf_{r\to\infty} \frac{logM(r,f)}{logr}.$$

Definition 3: Let $'a'$ be a complex number, finite or infinite. The Nevanlinna deficiency and the Valiron deficiency of $'a'$ with respect to a meromorphic function f are defined as

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

and

$$\Delta(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \limsup_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}.$$

Definition 4: The quantity $\Theta(a;f)$ of a meromorphic function f is defined as follows

$$\Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

Definition 5: [15] For $a\epsilon\mathbb{C} \cup \{\infty\}$we denote by$n(r,a;f| = 1)$, the number of simple zeros of $f - a$ in $|z| \le r$, $N(r,a;f| = 1)$ is defined in terms of $n(r,a;f| = 1)$in the usual way. We put

$$\delta_1(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f| = 1)}{T(r,f)},$$

the deficiency of $'a'$ corresponding to the simple a -points of f i.e., simple zeros of $f - a$.

Yang [14] proved that there exists at most a denumerable number of complex numbers $a\epsilon\mathbb{C} \cup \{\infty\}$ for which

$\delta_1(a;f) > 0$ and $\sum_{a\in C\cup\{\infty\}} \delta_1(a;f) \le 4$.

Definition 6: [9] For $a \in \mathbb{C}\cup\{\infty\}$ let $n_p(r,a;f)$ denotes the number of zeros of $f - a$ in $|z| \le r$ where a zero of "multiplicity $< p$□is counted according to its multiplicity and a zero of multiplicity $\ge p$ is counted exactly p times; and $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way". We define

$$\delta_p(a;f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,a;f)}{T(r,f)}.$$

Definition 7: [3] $P[f]$ is said to be admissible if

(i)$P[f]$ is homogeneous, or

(ii)$P[f]$ is non homogeneous and $m(r,f) = S(r,f)$.

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 1: [1] If f is meromorphic and g is entire then for all sufficiently large values of r,

$$T(r,fog) \le \{1+o(1)\}\frac{T(r,g)}{logM(r,g)}T(M(r,g),f).$$

Lemma 2: [2] Let f be meromorphic and g be entire and suppose that$0<\mu<\rho_g \le \infty$. Then for a sequence of values of r tending to infinity,

$$T(r,fog) \ge T(exp(r^{\mu}),f).$$

Lemma 3: [5] If f be any meromorphic function of order zero. Then

$$\lim_{r\to\infty}\frac{logT(r,f)}{log^{[2]}r}=1.$$

Lemma 4: [3] Let $P_0[f]$ be admissible. If f is of finite order or of non-zero lower order and $\sum_{a\neq\infty}\Theta(a;f)=2$ then

$$\lim_{r\to\infty}\frac{T(r,P_0[f])}{T(r,f)}=\Gamma_{P_0[f]}.$$

Lemma 5: [3] Let f be either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty,f)=\sum_{a\neq\infty}\delta(a;f)=1$ Then for homogeneous $P_0[f]$,

$$\lim_{r\to\infty}\frac{T(r,P_0[f])}{T(r,f)}=\gamma_{P_0[f]}.$$

Lemma 6: Let f be a meromorphic function of finite order or of non zero lower order. If $\sum_{a\neq\infty}\Theta(a;f)=2$, then the order (lower order) of homogeneous $P_0[f]$ is same as that of f if f is of positive finite order.

Proof: By Lemma 4

$$\lim_{r\to\infty}\frac{logT(r,P_0[f])}{logT(r,f)}$$

exists and is equal to 1.

$$\begin{aligned}\rho_{P_0[f]} &= \limsup_{r\to\infty}\frac{logT(r,P_0[f])}{logr}\\ &= \limsup_{r\to\infty}\frac{logT(r,f)}{logr}\lim_{r\to\infty}\frac{logT(r,P_0[f])}{logT(r,f)}\\ &= \rho_f .1=\rho_f .\end{aligned}$$

In a similar manner, $\lambda_{P_0[f]}=\lambda_f$.

This proves the lemma.

Lemma 7: Let f be a meromorphic function of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$.Then the order (lower order) of homogenous $P_0[f]$ and f are same when f is of finite positive order.

We omit the proof of the lemma because it can be carried out in the line of lemma 7 and with the help of Lemma 6.

In a similar manner we can state the following lemma without proof.

Lemma 8: Let f be a meromorphic function of finite order or of non-zero lower order such that $\delta(\infty,f)=\sum_{a\neq\infty}\delta(a;f)=1$. Then for every homogenous $P_0[f]$ the order (lower order) of $P_0[f]$ is same as that of f when f is of finite positive order.

Lemma 9: [10] Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\epsilon\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, Then

$$\lim_{r\to\infty}\frac{T(r,M[f])}{T(r,f)}=\Gamma_M-(\Gamma_M-\gamma_M)\Theta(\infty;f),$$

where $\Theta(\infty;f)=1-\limsup_{r\to\infty}\frac{\bar{N}(r,f)}{T(r,f)}$.

Lemma 10: If f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\epsilon\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, then the order and lower order of $M[f]$ are same as those of f when f is of finite positive order.

We omit the proof of the lemma because it can be carried out in the line of lemma 6 and with the help of Lemma 9.

3. THEOREMS

In this section we present the main results of the paper.

Theorem 1: Let f be a meromorphic function and g be an entire function such that $0<\lambda_f\le\rho_f<\rho_g<\infty$. Also let $\sum_{a\neq\infty}\Theta(a;f)=2$. Then

$$\limsup_{r\to\infty}\frac{logT(r,fog)}{T(r,P_0[f])}=\infty.$$

Proof: Since $\rho_f<\rho_g$ we can choose ε (>0) in such a way that

$$\rho_f+\varepsilon<\rho_g-\varepsilon<\rho_g. \tag{1}$$

Now in view of lemma 2 we obtain for a sequence of values of r tending to infinity that

$$logT(r,fog)\ge logT\{expr^{(\rho_g-\varepsilon)},f\}$$

$$i.e.,\ logT(r,fog)\ge(\lambda_f-\varepsilon)logexpr^{(\rho_g-\varepsilon)}$$

$$i.e.,\ logT(r,fog)\ge(\lambda_f-\varepsilon)\,r^{(\rho_g-\varepsilon)}. \tag{2}$$

Again by lemma 6, we have for all sufficiently large values of$r$,

$$logT(r,P_0[f])\le(\rho_{P_0[f]}+\varepsilon)logr$$

$$i.e.,logT(r,P_0[f])\le(\rho_f+\varepsilon)logr \tag{3}$$

$$i.e.,T(r,P_0[f])\le r^{(\rho_f+\varepsilon)}. \tag{4}$$

Therefore from (2) and (4) it follows for a sequence of values of r tending to infinity,

$$\frac{logT(r,fog)}{T(r,P_0[f])}\ge\frac{(\lambda_f-\varepsilon)r^{(\rho_g-\varepsilon)}}{r^{(\rho_f+\varepsilon)}}. \tag{5}$$

Now in view of (1) it follows from (5) that

$$\limsup_{r\to\infty}\frac{logT(r,fog)}{T(r,P_0[f])}=\infty.$$

This proves the theorem.

In the line of Theorem 1 the following corollary may be deduced.

Corollary 1: Let f be a meromorphic function and g be an entire function with $0<\lambda_f\le\rho_f<\lambda_g<\infty$. Also let $\sum_{a\neq\infty}\Theta(a;f)=2$. Then

$$\limsup_{r\to\infty}\frac{logT(r,fog)}{T(r,P_0[f])}=\infty.$$

Remark 1: The conclusion of Theorem 1and Corollary 1 can also be drawn under the hypothesis $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ instead of $\sum_{a\neq\infty}\Theta(a;f)=2$.

In the line of Theorem 1 and with the help of Lemma 10 we may state the following theorem without proof.

Theorem 2: Let f be a meromorphic function and g be an entire function such that $0<\lambda_f\le\rho_f<\rho_g<\infty$. Also let $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Then

$$\limsup_{r\to\infty}\frac{logT(r,fog)}{T(r,M[f])}=\infty.$$

In view of theorem 2, the following corollary may also be deduced. Hence the proof is omitted.

Corollary 2: Let f be a meromorphic function and g be an entire function such that $0 < \lambda_f \leq \rho_f < \lambda_g < \infty$. Also let $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$. Then

$$\limsup_{r\to\infty} \frac{logT(r,fog)}{T(r,M[f])} = \infty.$$

Theorem 3: Let f be a meromorphic function such that $0 < \lambda_f \leq \rho_f < \infty$, $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$ and g be an entire function with finite order, then for every positive constant A and every real number α

$$\limsup_{r\to\infty} \frac{logT(r,fog)}{\{logT(r^A,P_0[f])\}^{1+\alpha}} = \infty.$$

Proof: Let us suppose that

$$0 < \varepsilon < \min\{\lambda_f, \lambda_g\}.$$

If α is such that$1+\alpha \leq 0$, then the theorem is trivial. So we suppose that $1+\alpha > 0$. Now from Lemma 2 we get for a sequence of values of r tending to infinity

$$logT(r,fog) \geq logT(expr^{(\rho_g-\varepsilon)},f)$$

$$i.e., \ logT(r,fog) \geq (\lambda_f-\varepsilon)logexpr^{(\rho_g-\varepsilon)}$$

$$i.e., \ logT(r,fog) \geq (\lambda_f-\varepsilon)r^{(\rho_g-\varepsilon)}. \tag{6}$$

Again from the definition of $\rho_{P_0[f]}$ it follows for all sufficiently large values of r

$$logT(r^A,P_0[f]) \leq (\rho_{P_0[f]}+\varepsilon)Alogr$$

$$logT(r^A,P_0[f]) \leq (\rho_f+\varepsilon)Alogr$$

$$i.e., \{logT(r^A,P_0[f])\}^{1+\alpha} \leq (\rho_f+\varepsilon)^{1+\alpha}A^{1+\alpha}(logr)^{1+\alpha}. \tag{7}$$

Now from (6) and (7) it follows for a sequence of values of r tending to infinity

$$\frac{logT(r,fog)}{\{logT(r^A,P_0[f])\}^{1+\alpha}} \geq \frac{(\lambda_f-\varepsilon)r^{(\rho_g-\varepsilon)}}{(\rho_f+\varepsilon)^{1+\alpha}A^{1+\alpha}(logr)^{1+\alpha}}$$

Since$\frac{r^{(\rho_g-\varepsilon)}}{(logr)^{1+\alpha}} \to \infty$ as $r\to\infty$, the theorem follows from above.

Remark 2: The conclusion of Theorem (3) can also deduced if we replace $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$ by $\Theta(\infty;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$or $\sum_{a\neq\infty} \Theta(a;f) = 2$ rspectivly.

Theorem 4: Let f be a meromorphic function such that $0 < \lambda_f \leq \rho_f < \infty$, $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ and g be an entire function with finite order, then for every positive constant A and every real number α

$$\limsup_{r\to\infty} \frac{logT(r,fog)}{\{logT(r^A,M[f])\}^{1+\alpha}} = \infty.$$

The proof of the theorem can be established in the line of Theorem 3 and with the help of Lemma 10 and therefore is omitted.

Remark 3: The conclusion of Theorem 3, Theorem 4 and Remark 2 can also deduced if we take g be an entire function with non zero lower order instead of "finite order".

In the line of Theorem 3 one may state the following theorem without proof.

Theorem 5: Let f be a meromorphic function and g be an entire function such that $0<\lambda_g\le\rho_g<\infty$, and $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ or $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$ or $\sum_{a\neq\infty}\Theta(a;g)=2$, then for every positive constant A and every real number α

$$\limsup_{r\to\infty}\frac{logT(r,fog)}{\{logT(r^A,P_0[f])\}^{1+\alpha}}=\infty.$$

In the line of Theorem 5 and with the help of Lemma 10 we may state the following theorem without proof.

Theorem 6: Let f be a meromorphic function and g be an entire function such that $0<\lambda_g\le\rho_g<\infty$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$, Then for every positive constant A and every real number α

$$\limsup_{r\to\infty}\frac{logT(r,fog)}{\{logT(r^A,M[g])\}^{1+\alpha}}=\infty.$$

Theorem 7: Let f be a meromorphic function and g be an entire function such that $0<\lambda_f\le\rho_f<\infty$, $\sum_{a\neq\infty}\Theta(a;f)=2$ and $0<\rho_g<\infty$.Then

$$\frac{\rho_g}{\rho_f}\le\limsup_{r\to\infty}\frac{log^{[2]}T(r,fog)}{logT(r,P_0[f])}\le\frac{\rho_g}{\lambda_f}.$$

Proof: Since$T(r,g)\le log^+M(r,g)$, we have from Lemma 1 for all sufficiently large values of r,

$$T(r,fog)\le\{1+o(1)\}T(M(r,g),f)$$

$$i.e.,logT(r,fog)\le(\rho_f+\varepsilon)logM(r,g)+O(1)$$

$$i.e.,log^{[2]}T(r,fog)\le log^{[2]}M(r,g)+O(1)$$

$$i.e.,log^{[2]}T(r,fog)\le(\rho_g+\varepsilon)logr+O(1). \tag{8}$$

Again from (6) we obtain for a sequence of values of r tending to infinity that

$$log^{[2]}T(r,fog)\ge(\rho_g-\varepsilon)logr+O(1). \tag{9}$$

Also from the definition of $\lambda_{P_0[f]}$ we have for all sufficiently large values of r,

$$logT(r,P_0[f])\ge(\lambda_{P_0[f]}-\varepsilon)logr$$

$$i.e.,logT(r,P_0[f])\ge(\lambda_f-\varepsilon)logr. \tag{10}$$

Therefore from (8) and (10) we have for all sufficiently large values of r that

$$\frac{log^{[2]}T(r,fog)}{logT(r,P_0[f])}\le\frac{(\rho_g+\varepsilon)logr+O(1)}{(\lambda_f-\varepsilon)logr}$$

$$\limsup_{r\to\infty}\frac{log^{[2]}T(r,fog)}{logT(r,P_0[f])}\le\frac{\rho_g}{\lambda_f}. \tag{11}$$

Again from (3) and (9) it follows for a sequence of values of r tending to infinity

$$\frac{log^{[2]}T(r,fog)}{logT(r,P_0[f])}\ge\frac{(\rho_g-\varepsilon)logr+O(1)}{(\rho_f+\varepsilon)logr}$$

$$\limsup_{r\to\infty}\frac{log^{[2]}T(r,fog)}{logT(r,P_0[f])}\ge\frac{\rho_g}{\rho_f}. \tag{12}$$

Thus the theorem follows from (11) and (12).

Remark 4: In addition to the conditions of Theorem 7 if f is of regular growth i.e., $\rho_f=\lambda_f$. Then

$$\limsup_{r\to\infty}\frac{log^{[2]}T(r,fog)}{logT(r,P_0[f])}=\frac{\rho_g}{\rho_f}.$$

Remark 5: The conclusion of Theorem 7 and Remark 4 can also be drawn under the hypothesis $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ instead of $\sum_{a\neq\infty}\Theta(a;f)=2$.

Theorem 8: Let f be a meromorphic function and g be an entire function such that $0<\lambda_f\leq\rho_f<\infty$, $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ and $0<\rho_g<\infty$.Then

$$\frac{\rho_g}{\rho_f}\leq\limsup_{r\to\infty}\frac{log^{[2]}T(r,fog)}{logT(r,M[f])}\leq\frac{\rho_g}{\lambda_f}.$$

The proof is omitted because it can be carried out in the line of Theorem 7 and with the help of Lemma 10.

Remark 6: In addition to the conditions of theorem 8, let f is of regular growth, i.e., $\rho_f=\lambda_f$. Then

$$\limsup_{r\to\infty}\frac{log^{[2]}T(r,fog)}{logT(r,M[f])}=\frac{\rho_g}{\rho_f}.$$

In the line of Theorem 7 the following two corollaries may be deduced:

Corollary 3: Let f be a meromorphic function and g be an entire function such that $0<\lambda_f\leq\rho_f<\infty$, $0<\lambda_g\leq\rho_g<\infty$, and $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ or $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$ or $\sum_{a\neq\infty}\Theta(a;g)=2$. Then

$$\frac{\rho_g}{\rho_f}\leq\limsup_{r\to\infty}\frac{log^{[2]}T(r,fog)}{logT(r,P_0[g])}\leq\frac{\rho_g}{\lambda_f}.$$

In addition if g is of regular growth then

$$\limsup_{r\to\infty}\frac{log^{[2]}T(r,fog)}{logT(r,P_0[g])}=1.$$

Corollary 4: Let f be a meromorphic function and g be an entire function such that $0<\lambda_f\leq\rho_f<\infty$, $0<\lambda_g\leq\rho_g<\infty$, and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Then

$$\frac{\rho_g}{\rho_f}\leq\limsup_{r\to\infty}\frac{log^{[2]}T(r,fog)}{logT(r,M[g])}\leq\frac{\rho_g}{\lambda_f}.$$

In addition if g is of regular growth then

$$\limsup_{r\to\infty}\frac{log^{[2]}T(r,fog)}{logT(r,M[g])}=1.$$

Theorem 9: Let f be a meromorphic function of order zero and g be an entire function of non zero finite order. Also let $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$. Then

$$\limsup_{r\to\infty}\frac{logT(r,fog)}{logT(r^A,P_0[g])}\geq\frac{1}{A}$$

Where $A>0$.

Proof: In view of Lemma 2 and Lemma 3 we obtain for a sequence of values of r tending to infinity that

$$logT(r,fog)\geq logT\{expr^{(\rho_g-\varepsilon)},f\}$$

$$i.e., logT(r,fog)\geq(1-\varepsilon)log^{[2]}expr^{(\rho_g-\varepsilon)}$$

$$i.e., logT(r,fog)\geq(1-\varepsilon)(\rho_g-\varepsilon)logr\ . \tag{13}$$

Again by Lemma 7, we have for all sufficiently large values of r,

$$logT(r^A,P_0[g])\leq(\rho_{P_0[g]}+\varepsilon)logr^A$$

$$i.e., logT(r^A,P_0[g])\leq A(\rho_g+\varepsilon)logr. \tag{14}$$

Therefore from (13) and (14) we obtain for a sequence of values of r tending to infinity that

$$\frac{logT(r,fog)}{logT(r^A,P_0[g])}\geq\frac{(1-\varepsilon)(\rho_g-\varepsilon)logr}{A(\rho_g+\varepsilon)logr}. \tag{15}$$

Since $\varepsilon(>0)$ is arbitrary, the theorem follows from (15).

Remark 7: The conclusion of Theorem 9 can also deduce if we replace $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ by $\sum_{a\neq\infty}\Theta(a;g)=2$ or $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$ respectively.

Theorem 10: Let f be a meromorphic function of order zero and g be an entire function of non zero finite order. Also let $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Then for any positive real number A.

$$\limsup_{r\to\infty}\frac{logT(r,fog)}{logT(r^A,M[g])}\geq\frac{1}{A}.$$

The proof is omitted because it can be carried out in the line of Theorem 9 and with the help of Lemma 10.

Theorem 11: Let f and g be two entire functions such that $0<\lambda_f^{**}<\infty$, $0<\lambda_g\leq\rho_g<\infty$ and $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$. Then

$$\lim_{r\to\infty}\frac{logT(r,fog)}{logT(r^A,P_0[g])}=\infty,$$

where A is any positive real number.

Proof: We know that for each $r>0$ [11]

$$T(r,fog)\geq\frac{1}{3}logM\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o(1),f\right\} \tag{16}$$

Let us choose ε in such a way that $0<\varepsilon<\min\{\lambda_f^{**},\lambda_g\}$.

Now we get from (16) for all sufficiently large values of r that

$$T(r,fog)\geq\frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)logM\left(\frac{r}{4},g\right)+O(1)$$

$$i.e.,T(r,fog)\geq\frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\left(\frac{r}{4}\right)^{\lambda_g-\epsilon}+O(1). \tag{17}$$

Therefore we obtain from (14) and (17) for all sufficiently large values of r that

$$\frac{T(r,fog)}{logT\left(r^A,P_0[g]\right)}\geq\frac{\frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\left(\frac{r}{4}\right)^{\lambda_g-\epsilon}+O(1)}{A(\rho_g+\varepsilon)logr}. \tag{18}$$

As $\lambda_g>0$, the theorem follows from (18).

Remark 8: If we take $0<\rho_f^{**}<\infty$ instead of $0<\lambda_f^{**}<\infty$ in Theorem 11and the other conditions remain the same, then in the line of Theorem 11 one can easily verify that

$$\limsup_{r\to\infty}\frac{T(r,fog)}{logT(r^A,P_0[g])}=\infty.$$

Remark 9: Also if we consider $0<\lambda_g<\infty$ or $0<\rho_g<\infty$ instead of $0<\lambda_g\leq\rho_g<\infty$ in Theorem 11 and the other conditions remain the same, then in the line of Theorem 11 one can easily verify that

$$\limsup_{r\to\infty}\frac{logT(r,fog)}{logT(r^A,P_0[g])}=\infty.$$

Remark 10: The conclusion of Theorem 11, Remark 8 and Remark 9 can also deduced if we replace $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ by $\sum_{a\neq\infty}\Theta(a;g)=2$ or $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$ respectively.

In the line of Theorem 11 and with the help of Lemma 10 we may state the following theorem without proof.

Theorem 12: Let f and g be two entire functions such that $0<\lambda_f^{**}<\infty$, $0<\lambda_g\leq\rho_g<\infty$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Then

$$\lim_{r\to\infty}\frac{T(r,fog)}{logT(r^A,M[g])}=\infty,$$

where A is any real number.

Remark 11: If we take $0<\rho_f^{**}<\infty$ instead of $0<\lambda_f^{**}<\infty$ in Theorem 12 and the other conditions remain the same, then in the line of Theorem 12 one can easily verify that

$$\lim_{r\to\infty}\frac{T(r,fog)}{logT(r^A,M[g])}=\infty.$$

Remark 12: Also if we consider $0 < \lambda_g < \infty$ or $0 < \rho_g < \infty$ instead of $0 < \lambda_g \le \rho_g < \infty$ in Theorem 12 and the other conditions remain the same, then in the line of Theorem12 it can be shown that

$$\limsup_{r\to\infty} \frac{T(r, fog)}{logT(r^A, M[g])} = \infty.$$

Remark 13 : If we take f be a meromorphic function with order zero in Theorem 11and Theorem 12 and the other conditions remain the same then Theorem 11and Theorem 12 remain valid with "limsup" instead of "lim".

Theorem 13: Let f be meromorphic and g be entire functions such that $0 < \lambda_f \le \rho_f < \infty$, $\sum_{a\neq\infty} \Theta(a; f) = 2$ and $\rho_g^{**} < \infty$. Then

$$\lim_{r\to\infty} \frac{logT(r, fog)}{T(r^A, P_0[f])} = 0,$$

where A is any positive real number.

Proof: In view of Lemma 1 and the inequality $T(r, g) \le log^+M(r, g)$ we get for all sufficiently large values of$r$ that,

$$T(r, fog) \le \{1 + o(1)\}T(M(r, g), f)$$

$$i.e., logT(r, fog) \le (\rho_f + \varepsilon)logM(r, g) + O(1)$$

$$i.e., logT(r, fog) \le (\rho_f + \varepsilon)(\rho_g^{**} + \varepsilon)logr + O(1). \quad (19)$$

Again from the definition of $\lambda_{P_0[f]}$ we have for arbitrary positive ε and for all sufficiently large values of r,

$$logT(r^A, P_0[f]) \ge A(\lambda_{P_0[f]} - \varepsilon)logr$$

$$i.e., T(r^A, P_0[f]) \ge r^{A(\lambda_f - \varepsilon)}. \quad (20)$$

Therefore it follows from (19) and (20) for all sufficiently large values of r that

$$\frac{logT(r, fog)}{T(r^A, P_0[f])} \le \frac{(\rho_f+\varepsilon)(\rho_g^{**}+\varepsilon)logr + O(1)}{r^{A(\lambda_f-\varepsilon)}}. \quad (21)$$

As $\lambda_f > 0$, the theorem follows from (21).

Remark 14: If we take $0 < \rho_f < \infty$ or $0 < \lambda_f < \infty$ instead of $0 < \lambda_f \le \rho_f < \infty$ in Theorem 13 and the other conditions remain the same, then in the line of Theorem 13 one can easily verify that

$$\liminf_{r\to\infty} \frac{logT(r, fog)}{T(r^A, P_0[f])} = 0.$$

Remark 15: Also if we take $\lambda_g^{**} < \infty$ instead of $\rho_g^{**} < \infty$ in Theorem 13 and the other conditions remain the same, then in the line of Theorem 13 one can easily verify that

$$\liminf_{r\to\infty} \frac{logT(r, fog)}{T(r^A, P_0[f])} = 0.$$

Remark 16: The conclusion of Theorem 13, Remark 14 and Remark 15 can also deduced if we replace $\sum_{a\neq\infty} \Theta(a; f) = 2$ by $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ respectively.

In the line of Theorem 13 and with the help of Lemma 10 we may state the following theorem without proof.

Theorem 14: Let f be meromorphic and g be entire functions such that $0 < \lambda_f \le \rho_f < \infty$, $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$ and $\rho_g^{**} < \infty$. Then for any positive real number A,

$$\lim_{r\to\infty} \frac{logT(r, fog)}{T(r^A, M[f])} = 0.$$

Remark 17: If we take $0 < \rho_f < \infty$ or $0 < \lambda_f < \infty$ instead of $0 < \lambda_f \le \rho_f < \infty$ in Theorem 14 and the other conditions remain the same, then in the line of Theorem 14 one can easily verify that

$$\liminf_{r\to\infty} \frac{logT(r, fog)}{T(r^A, M[f])} = 0.$$

Remark 18: Also if we take $\lambda_g^{**} < \infty$ instead of $\rho_g^{**} < \infty$ in Theorem 14 and the other conditions remain the same, then in the line of Theorem 14 one can easily verify that

$$\liminf_{r\to\infty} \frac{logT(r, fog)}{T(r^A, M[f])} = 0,$$

where A is any real number.

REFERENCES

[1] Bergweiler, W.: On the Nevanlinna characteristic of a composite function, Complex Variables, Vol.10 (1988), pp. 225- 236.

[2] Bergweiler, W.: On the growth rate of composite meromorphic functions, Complex Variables, Vol.14 (1990), pp. 187- 196.

[3] Bhattacharjee, N. and Lahiri, I.: Growth and value distribution of differential polynomials, Bull. Math. Soc. Sc. Math. Roumanie Tome, Vol.39 (87), No. 1- 4(1996), pp. 85- 104.

[4] Clunie, J.: The Composition of Entire and Meromorphic Functions, Mathematical essays dedicated to A.J. Macintyre, Ohio University Press, 1970, pp. 75- 92.

[5] Datta, S.K and Biswas, T.: On the definition of a meromorphic function of order zero, Int. Math. Forum, Vol.4, No. 37(2009), pp.1851-1861.

[6] Doeringer, W.: Exceptional values of differential polynomials, Pacific J. Math., Vol. 98, No. 1(1982), pp. 55- 62.

[7] Hayman, W.K.: Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[8] Lahiri, I.: Growth of composite integral functions, Indian J. Pure Appl. Math., Vol. 20, No.9,(September 1989), pp. 899- 907.

[9] Lahiri, I.: Deficiencies of differential polynomials, Indian J. Pure Appl. Math., Vol. 30, No. 5(1999), pp. 435- 447.

[10] Lahiri, I. and Datta, S.K.: Growth and value distribution of differential monomials, Indian J. Pure Appl. Math., Vol.32, No12, December (2001), pp. 1831- 1841.

[11] Niino, K. and Yang, C.C. : Some growth relationships on factors of two composite entire functions, Factorization theory of meromorphic functions and related topics, Marcel Dekker Inc.(New York and Basel), (1982), pp. 95-99.

[12] Sons, L. R. : Deficiencies of monomials, Math. Z, Vol. 111(1969), pp. 53- 68.

[13] Singh, A.P.: Growth of composite entire functions, Kodai Math. J., Vol. 8(1985), pp.99- 102.

[14] Yang, L.: Value distribution theory and new research on it, Science Press, Beijing (1982).

[15] Yi, H.X.: On a result of Singh, Bull. Austral. Math. Soc., Vol. 41(1990), pp. 417- 420.

[16] Valiron, G.: Lectures on the General Theory of Integral Functions, Chelsea Publishing Company, 1949.

Source of support: Nil, Conflict of interest: None Declared