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ON SOME PROPERTIES OF CLIQUISH FUNCTIONS

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ABSTRACT

In this paper, some interesting properties of cliquish functions are investigated. It is established that the set of all real valued bounded cliquish functions on a topological space forms a commutative Banach algebra under the supremum norm and is shown that the space of all real valued Quasicontinuous functions is a closed subalgebra of this Banach algebra.

Key Words: *Cliquish function, Banach algebra, Quasicontinuity, Semi-continuity.*

AMS subject classification: *54C08, 54C30.*

INTRODUCTION

This paper is devoted to the properties of cliquish functions and algebraic structures generated by these functions. It is proved that the set of all real valued bounded cliquish functions defined on a topological space forms a commutative Banach algebra with identity under supremum norm.

In what follows $\mathcal{X}$ and $\mathcal{N}$ stand for a topological space and a normed linear space respectively. Let $\mathbb{R}$ denote the set of all real numbers.

1. PRELIMINARIES

The Definitions that are needed throughout this paper are presented in this section.

1.1 Definition [1]: A function $f:\mathcal{X}\to\mathcal{N}$ is said to be cliquish at a point $x\in\mathcal{X}$ if for every $\varepsilon>0$ and every neighborhood U of x in $\mathcal{X}$ there exists a non-empty open set $W\subset U$ such that

$$\|f(y)-f(z)\|<\varepsilon \quad \forall \; y,z\in W.$$

If f is cliquish at every point of $\mathcal{X}$ then we say that f is cliquish on $\mathcal{X}$.

1.2 Definition [3]: A function $f:\mathcal{X}\to\mathcal{N}$ is said to be semi-continuous at a point $x\in\mathcal{X}$ if for every neighborhood U of x and every $\varepsilon>0$ there exists a non-empty open set $W\subseteq U$ such that

$$\|f(y)-f(x)\|<\varepsilon \quad \forall \; y\in W.$$

If $f:\mathcal{X}\to\mathcal{N}$ is semi-continuous at every point of $\mathcal{X}$ then we say that f is semi-continuous on $\mathcal{X}$.

1.3 Definition: A function $f:\mathcal{X}\to\mathcal{N}$ is said to be Quasicontinuous at a point $p\in\mathcal{X}$ if both $f(p+)$ and $f(p-)$ exist. We say that f is Quasicontinuous on $\mathcal{X}$ if f is Quasicontinuous at every point of X.

***1.4* Definition:** A function $f:\mathcal{X}\to\mathcal{N}$ is said to be $^{+}$Quasicontinuous at a point $p\in\mathcal{X}$ if $f(p+)$ exists. We say that f is $^{+}$Quasicontinuous on X if f is $^{+}$Quasicontinuous at every point of $\mathcal{X}$.

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1.5 Definition: A function $f:\mathcal{X}\to\mathcal{N}$ is said to be $^{-}$Quasicontinuous at a point $p\in\mathcal{X}$ if $f(p-)$ exists. We say that f is $^{-}$Quasicontinuous on $\mathbb{R}$ if f is $^{-}$Quasicontinuous at every point of $\mathbb{R}$.

1.6 Remarks:
(i) The set $\mathscr{Q}(I)$ of all Quasicontinuous functions $f:I\to\mathbb{R}$, where I is any closed and bounded interval on $\mathbb{R}$ is a commutative Banach algebra with identity under the supremum norm.

(ii) The set $\mathscr{Q}^{+}(\mathbb{R})$ of all bounded $^{+}$Quasicontinuous real functions defined on $\mathbb{R}$ forms a commutative Banach algebra with identity under the supremum norm.

(iii) The set $\mathscr{Q}^{-}(\mathbb{R})$ of all bounded $^{-}$Quasicontinuous real functions defined on $\mathbb{R}$ forms a commutative Banach algebra with identity under the supremum norm.

(iv) $\mathscr{Q}^{+}(\mathbb{R})\bigcap\mathscr{Q}^{-}(\mathbb{R})=\mathscr{Q}(\mathbb{R})$.

2. PROPERTIES OF CLIQUISH FUNCTIONS

2.1 Proposition: If $f:\mathcal{X}\to\mathcal{N}$ and $g:\mathcal{X}\to\mathcal{N}$ are cliquish on $\mathcal{X}$ then $f+g$ is cliquish on $\mathcal{X}$.

Proof: Let $x_0\in\mathcal{X}$ and let $\varepsilon>0$ be given. Let U be a neighborhood of x_0 in $\mathcal{X}$. Then there exists a non-empty open set $V\subset U$ such that $\|f(x)-f(y)\|<\frac{\varepsilon}{2}\ \ \forall\ x,y\in V$.

Choose a point y_0 in V. Since g is cliquish at y_0, there exists a non-empty open set $W\subset V$ such that $\|g(x)-g(y)\|<\frac{\varepsilon}{2}\ \ \forall\ x,y\in W$.

$$x,y\in W\ \Rightarrow\ \|(f+g)(x)-(f+g)(y)\|\le\|f(x)-f(y)\|+\|g(x)-g(y)\|$$
$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus for every $\varepsilon>0$ and every neighborhood U of x_0, there exists a non-empty open set $W\subset U$ such that

$$\|(f+g)(x)-(f+g)(y)\|<\varepsilon\quad\forall\ x,y\in W$$

$\Rightarrow$ $f+g$ is cliquish at x_0.

2.2 Remark: From the following example it is clear that if f and g are cliquish at a point x_0 then it is not necessary that $f+g$ is cliquish at x_0.

2.3 Example: Define $\mathsf{f}:[-1,1]\to\mathbb{R}$ and $\mathsf{g}:[-1,1]\to\mathbb{R}$ as follows.

$$\mathsf{f}(x)=\begin{cases}\frac{1}{x} & \text{if}\quad 0<x\le 1\\ 0 & \text{if}\quad -1\le x\le 0\end{cases}$$

$$\mathsf{g}(x)=\begin{cases}\frac{1}{x} & \text{if}\quad -1\le x<0\\ 0 & \text{if}\quad 0\le x\le 1\end{cases}$$

Then $(\mathrm{f}+\mathrm{g})\ (x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$

Clearly f and g are cliquish at $x = 0$, but $f + g$ is not cliquish at $x = 0$.

2.3 Proposition: If $c \in \mathbb{R}$ and $f : \mathcal{X} \to \mathcal{N}$ is cliquish at a point $x_0 \in \mathcal{X}$ then cf is cliquish at x_0.

Proof: If $c = 0$ then $cf = O$, where O: $\mathcal{X} \to \mathcal{N}$ is defined by $O(x) = 0 \quad \forall \quad x \in \mathcal{X}$. Then cf is cliquish on $\mathcal{X}$.

Now suppose that $c \neq 0$ .Let $\varepsilon > 0$ be given and let U be a neighborhood of x_0 in $\mathcal{X}$.

Then there exists a non-empty open set $W \subset U$ such that $\| f(x) - f(y) \| < \dfrac{\varepsilon}{|c|} \quad \forall \quad x, y \in W$.

$\Rightarrow \quad \|(cf)(x) - (cf)(y)\| < \varepsilon \quad \forall \quad x, y \in W$.

Hence cf is cliquish at x_0.

2.4 Remark: A real valued cliquish function defined on a compact space is not necessarily bounded. Also a bounded function need not be cliquish. For example, the characteristic function $\xi_{\mathbb{Q}}$ of rationals is bounded, but not cliquish.

The function $h : [-1,1] \to \mathbb{R}$ defined by $h(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad 0 < x \leq 1 \\ 2 & \text{if} \quad -1 \leq x \leq 0 \end{cases}$ cliquish, but not bounded.

2.5 Proposition: If $f : \mathcal{X} \to \mathcal{N}$ and $g : \mathcal{X} \to \mathcal{N}$ are bounded and cliquish on $\mathcal{X}$ then so is fg.

Proof: Let $x_0 \in \mathcal{X}$ and $\varepsilon > 0$ be given.

If U is any neighborhood of x_0 in $\mathcal{X}$ then there exists a non-empty open set $W \subset U$ such that $\| f(y) - f(z) \| < \varepsilon$ and $\| g(y) - g(z) \| < \varepsilon \quad \forall \quad y, z \in W$.

Put $M = \sup\{\| f(x) \| / x \in \mathcal{X}\}$ and $K = \sup\{\| g(x) \| / x \in \mathcal{X}\}$.

For $y \in W$ and $z \in W$, we have

$$\begin{aligned} \|(fg)(y) - (fg)(z)\| &= \| f(y)g(y) - f(z)g(z) \| \\ &= \| f(y)g(y) - f(y)g(z) + f(y)g(z) - f(z)g(z) \| \\ &\leq \| f(y) \| \, \| g(y) - g(z) \| + \| g(z) \| \, \| f(y) - f(z) \| \\ &< (M + K)\varepsilon. \end{aligned}$$

Hence fg is cliquish at x_0.

Since f and g are bounded, fg is bounded on $\mathcal{X}$.

Hence fg is bounded and cliquish on $\mathcal{X}$.

2.6 Notation : The symbol $\mathcal{K}(\mathcal{X})$ denotes the set of all real valued bounded cliquish functions defined on $\mathcal{X}$.

2.7 Proposition: If $f \in \mathcal{K}(\mathcal{X})$ and $g \in \mathcal{K}(\mathcal{X})$ then

(a) $f \vee g \in \mathcal{K}(\mathcal{X})$

(b) $f \wedge g \in \mathcal{K}(\mathcal{X})$ where $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$.

Proof: (a) Let $\varepsilon > 0$ be given and P in $\mathcal{X}$ $p \in \mathbb{R}$. Let U be a neighborhood of p in $\mathbb{R}$.

Since f is cliquish at p, there exists a non-empty open set $G \subset U$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2} \quad \forall \ x, y \in G.$$

Choose a point $q \in G$. Since g is cliquish at q, there exists a non-empty open set $H \subset G$ such that $|g(x) - g(y)| < \frac{\varepsilon}{2} \quad \forall \ x, y \in H$.

Suppose that $x \in H$ and $y \in H$.

$$\Rightarrow \ |f(x) - f(y)| < \frac{\varepsilon}{2}$$

$$\Rightarrow \ -\frac{\varepsilon}{2} < f(x) - f(y) < \frac{\varepsilon}{2}$$

$$\Rightarrow \ f(y) < f(x) + \frac{\varepsilon}{2} \quad \text{and} \quad f(x) < \frac{\varepsilon}{2} + f(y)$$

$$\Rightarrow \ f(y) < (f \vee g)(x) + \frac{\varepsilon}{2} \quad \text{and} \quad f(x) < \frac{\varepsilon}{2} + (f \vee g)(y)$$

Similarly $g(y) < (f \vee g)(x) + \frac{\varepsilon}{2}$ and $g(x) < \frac{\varepsilon}{2} + (f \vee g)(y)$.

Hence $(f \vee g)(y) \le (f \vee g)(x) + \frac{\varepsilon}{2}$ and $(f \vee g)(x) \le (f \vee g)(y) + \frac{\varepsilon}{2}$

$$\Rightarrow |(f \vee g)(x) - (f \vee g)(y)| \le \frac{\varepsilon}{2}$$

Thus for every $\varepsilon > 0$ and every neighborhood U of p, there exists a non-empty open set $H \subset U$ such that

$$|(f \vee g)(x) - (f \vee g)(y)| < \varepsilon \quad \forall \ x, y \in H$$

$\Rightarrow$ $f \vee g$ is cliquish at p.

This completes the proof of (a).

The proof of (b) follows from the fact that $f \wedge g = -((-f) \vee (-g))$ and from (a).

2.8 Proposition: If $f_n : \mathcal{X} \to \mathcal{N}$, $n = 1, 2, 3, \ldots,$ is cliquish at a point $x \in \mathcal{X}$ and $f_n \to f$ uniformly on $\mathcal{X}$ then f is cliquish at x.

Proof: Let $\varepsilon > 0$ be given and let U be a neighborhood of x in $\mathcal{X}$.

Since $f_n \to f$ uniformly on X, there exists an integer N such that

$$n \ge N \Rightarrow \|f_n(y) - f(y)\| < \frac{\varepsilon}{3} \quad \forall \ y \in \mathcal{X}.$$

Since f_N is cliquish at x, there exists a non-empty open set $W \subset U$ such that

$$\| f_N(y) - f_N(z) \| < \frac{\varepsilon}{3} \quad \forall \; y, z \in W.$$

$$y, z \in W \;\Rightarrow\; \| f(y) - f(z) \| \le \| f(y) - f_N(y) \| + \| f_N(y) - f_N(z) \| + \| f_N(z) - f(z) \|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence f is cliquish at x.

2.9 Proposition: The set $\mathcal{K}(\mathcal{X})$ forms a real commutative Banach algebra with identity under supremum norm. The identity is given by the function which is identically 1.

2.10 Proposition: The set $\mathcal{K}(\mathcal{X})$ is a lattice with respect to the relation $\le$ defined by

$$f \le g \Leftrightarrow f(x) \le g(x) \quad \forall \; x \in \mathcal{X}.$$

3. ALGEBRAIC STRUCTURES

3.1 Definition: Let $\mathcal{K}$ be the commutative Banach algebra of all bounded cliquish functions $f : \mathbb{R} \to \mathbb{R}$, with identity under the sup norm. We define the following.

$\mathcal{A}(\mathcal{K}) = \{ f : \mathbb{R} \to \mathbb{R} \,/\, f$ is the finite sum of functions from $\mathcal{K}\}$

$\mathcal{P}(\mathcal{K}) = \{ f : \mathbb{R} \to \mathbb{R} \,/\, f$ is the finite product of function from $\mathcal{K}\}$

3.2 Proposition: $\mathcal{P}(\mathcal{K}) = \mathcal{K}$.

3.3 Proposition: $\mathcal{A}(\mathcal{K})$ is a normed linear subspace of $\mathcal{K}$.

4. POINTS OF CLIQUISHNESS

4.1 Definition: Let $f : \mathcal{X} \to \mathcal{N}$. We define $\mathcal{K}(f) = \{x \in \mathcal{X} / f$ is cliquish at $x\}$.

4.2 Proposition: Let $f : \mathcal{X} \to \mathcal{N}$. The set $\mathcal{K}(f)$ is closed in $\mathcal{X}$.

Proof: Let x be a limit point of $\mathcal{K}(f)$ in $\mathcal{X}$. Then there exists a sequence $\{x_n\}$ in $\mathcal{K}(f)$ such that $x_n \to x$ in $\mathcal{X}$. Let G be a neighborhood of x in $\mathcal{X}$. Then there exists an integer N such that $x_n \in G \quad \forall \; n > N$.

Since $x_n \in \mathcal{K}(f)$, there exists a non-empty open set $H \subset G$ such that

$$\| f(y) - f(z) \| < \varepsilon \quad \forall \; y, z \in H$$

$\Rightarrow$ f is cliquish at x

$\Rightarrow$ $x \in \mathcal{K}(f)$

Hence $\mathcal{K}(f)$ is closed in $\mathcal{X}$.

4.3 Proposition [3]: The set $\mathcal{K}(f) - C(f)$ is of first category in $\mathcal{X}$, where $C(f)$ is the points of continuity of f in $\mathcal{X}$.

4.4 Proposition: $\mathcal{X}$ is a Baire space if and only if every cliquish function $f : \mathcal{X} \to \mathbb{R}$ has a dense set of points of continuity.

5. RELATIONSHIP WITH QUASICONTINUOUS AND SEMI-CONTINUOUS FUNCTIONS

5.1 Proposition: Let $f:\mathbb{R}\to\mathbb{R}$ and $p\in\mathbb{R}$. If $f(p+)$ exists then f is cliquish at p.

Proof: Let $\varepsilon>0$ be given and let U be a neighborhood of p in $\mathbb{R}$. Then there exists a $\delta_1>0$ such that $(p-\delta_1, p+\delta_1)\subset U$.

Since $f(p+)$ exists, there exists $\delta_2>0$ such that

$|f(x)-f(p+)|<\dfrac{\varepsilon}{2} \quad \forall \; x\in(p, p+\delta_2)$.

Put $\delta=\min\{\delta_1,\delta_2\}$ and $W=(p, p+\delta)$.

Then for $x, y\in W$, we have
$$\begin{aligned}|f(x)-f(y)| &= |f(x)-f(p+)+f(p+)-f(y)|\\ &\le |f(x)-f(p+)| + |f(y)-f(p+)|\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Thus for every $\varepsilon>0$ and every neighborhood U of p, there exists a non-empty open set $W\subset U$ such that

$|f(x)-f(y)|<\varepsilon \quad \forall \; x, y\in W$

$\Rightarrow$ f is cliquish at p.

5.2 Proposition: Let $f:\mathbb{R}\to\mathbb{R}$ and $p\in\mathbb{R}$. If $f(p-)$ exists then f is cliquish at p.

5.3 Remark: From the above Propositions 5.1 and 5.2 it can be observed that (a) $\mathcal{Q}^{+}(\mathbb{R})\subset\mathcal{K}(\mathbb{R})$ (b) $\mathcal{Q}^{-}(\mathbb{R})\subset\mathcal{K}(\mathbb{R})$ (c) $\mathcal{Q}(\mathbb{R})\subset\mathcal{K}(\mathbb{R})$, where $\mathcal{K}(\mathbb{R})$ is the set of all bounded real valued cliquish functions defined on $\mathbb{R}$. From example 2.3 it can be easily seen that the converses of the above Propositions 5.1 and 5.2 are not true.

5.4 Proposition: If $f:\mathcal{X}\to\mathcal{N}$ *is* semi-continuous at a point $x_0\in\mathcal{X}$ then f is cliquish at x_0.

Proof: Let $f:\mathcal{X}\to\mathcal{N}$ *be* semi-continuous at a point $x_0\in\mathcal{X}$. Let $\varepsilon>0$ be given. Let U be a neighborhood of x_0 in $\mathcal{X}$.

Then there exists a non-empty open set G contained in U such that

$\|f(y)-f(x_0)\|<\dfrac{\varepsilon}{2} \quad \forall \; y\in G$

Let $y, z\in G$

$\Rightarrow \|f(y)-f(x_0)\|<\dfrac{\varepsilon}{2}$ and $\|f(z)-f(x_0)\|<\dfrac{\varepsilon}{2}$

$$\begin{aligned}\Rightarrow \|f(y)-f(z)\| &= \|f(y)-f(x_0)+f(x_0)-f(z)\|\\ &\le \|f(y)-f(x_0)\|+\|f(z)-f(x_0)\|\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Thus for every $\varepsilon > 0$ and every neighborhood U of x there exists a non-empty open set $G \subset U$ such that

$\| f(y) - f(z) \| < \varepsilon \quad \forall \ y, z \in G$

$\Rightarrow$ f is cliquish at x_0.

5.5 Remark: By the above Proposition 5.4 it is clear that every semi-continuous function from a topological space into a normed linear space is cliquish. The converse is not true as is evident from the following example.

5.6 Example: Define $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if} \quad -1 \le x < 0 \\ 1 & \text{if} \quad x = 0 \\ 2 & \text{if} \quad 0 < x \le 1 \end{cases}$$

Clearly f is cliquish at $x = 0$, but f is not semi-continuous at $x = 0$.

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