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SALAGEAN-TYPE HARMONIC UNIVALENT FUNCTIONS WITH FIXED POINTS

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ABSTRACT

The purpose of the present paper is to establish some results involving coefficient conditions, extreme points, distortion bounds, convex combination and radii of convexity for a new class of Salagean-type harmonic univalent functions fixed points in the open unit disc.

Keywords: *Harmonic, Univalent functions, Salagean derivative, Fixed points.*

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1. INTRODUCTION

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f.

A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that

$$|h'(z)| > |g'(z)|, \ z \in D.$$

Let S_H denotes the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , \ g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{1}$$

Clunie and Sheil-Small [2] investigated the class S_H as well as its geometric subclasses and established some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

For $f = h + \overline{g}$ given by (1), Jahangiri et al. [6] defined the modified Salagean operator of f as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)} \ , \qquad (m \in N_0, N_0 = N \cup \{0\}) \tag{2}$$

where

$$D^m h(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k \ \text{ and } \ D^m g(z) = \sum_{k=1}^{\infty} k^m b_k z^k ,$$

where D^m stands for the differential operator introduced by Salgean [9].

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Now for $0 \le \alpha < 1,\ 0 \le \lambda < 1, m \in N, n \in N_0, m > n$ and $z \in U$, suppose that $S_H(m,n;\alpha;\lambda)$ denote the family of harmonic functions f of the form (1) such that

$$\operatorname{Re}\left\{\frac{D^m f(z)}{\lambda D^m f(z) + (1-\lambda) D^n f(z)}\right\} > \alpha, \tag{3}$$

where $D^m f$ is defined by (2).

Further let the subclass $\overline{S}_H(m,n;\alpha;\lambda)$ consist of harmonic functions $f_m = h + \overline{g_m}$ in $\overline{S}_H(m,n;\alpha;\lambda)$ so that h and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k \text{ and } g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_k z^k;\ \ a_k, b_k \ge 0. \tag{4}$$

By specializing the parameters in subclass $S_H(m,n;\alpha;\lambda)$ we obtain the following known subclasses studied earlier by various authors.

1. If we put $\lambda = 0$ then it reduces to the class $S_H(m,n;\alpha)$ studied by Yalcin [13].

2. If we put $m=1, n=0, \lambda=0$ and $m=2, n=1, \lambda=0$ then it reduces to the class $HS(\alpha)$ and $HK(\alpha)$ studied by Jahangiri [5].

3. If we put $m=1, n=0, \alpha=0, \lambda=0$ and $m=2, n=1, \alpha=0, \lambda=0$ with $b_1=0$ then it reduces to the class $HS(0)$ and $HK(0)$ studied by Avci and Zlotkiewicz [1] and Silverman [11].

4. If we put $m=1, n=0, \alpha=0, \lambda=0$ and $m=2, n=1, \alpha=0, \lambda=0$ then it reduces to the class $HS(0)$ and $HK(0)$ studied by Silverman and Silvia [12], which is an improvement of [1, 11].

5. If we put $m = n+1, \lambda = 0$ then it reduces to the class $H(n,\alpha)$ studied by Jahangiri et al. [6].

6. If we put $m=1, n=0$ then it reduces to the class $S_H^*(\lambda,\alpha)$ studied by Öztürk et al. [8].

The classes $S_H(m,n;\alpha;\lambda)$ and $\overline{S}_H(m,n;\alpha;\lambda)$ were extensively studied by Dixit and Porwal [3].

Several authors, such as ([4], [7], [10]) studied the subclasses of analytic univalent functions with fixed points. Recently, Dixit and Porwal [6] investigated a subclass of harmonic univalent functions with fixed points and negative coefficients. In the present paper an attempt has been made to study the subclasses of Salagean-type harmonic univalent functions with fixed point in the following way

A function $f = h + \overline{g}$ where

$$h(z) = a_1 z + \sum_{k=2}^{\infty} a_k z^k,\ \ g(z) = \sum_{k=1}^{\infty} b_k z^k,\ \ a_1 > 0,\ |b_1| < 1 \tag{5}$$

is said to be in the family $S_H(m,n;\alpha;\lambda,z_0)$ if the coefficient condition (3) is satisfied

$$f(z_0) = z_0, -1 < z_0 < 1,\ z_0 \ne 0. \tag{6}$$

Further, we let $\overline{S_H}(m,n;\alpha;\lambda,z_0)$ consist of harmonic functions $f_m = h + \overline{g_m}$ is in $S_H(m,n;\alpha;\lambda,z_0)$ so that h and g_m are of the form

$$h(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k \text{ and } g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_k z^k;\ \ a_k, b_k \ge 0. \tag{7}$$

In the present paper, results involving the coefficients, extreme points, distortion bounds, convex combinations and radii of convexity for the above classes $S_H\left(m,n;\alpha;\lambda,z_0\right)$ and $\overline{S_H}\left(m,n;\alpha;\lambda,z_0\right)$ of harmonic univalent functions have been investigated.

2. MAIN RESULTS

We first prove a necessary and sufficient condition for functions in $\overline{S_H}\left(m,n;\alpha;\lambda,z_0\right)$.

Theorem 2.1: Let $f_m = h+\overline{g_m}$ be such that *h* and g_m are given by (7). Furthermore, let

$$\sum_{k=2}^{\infty}\frac{k^m\left(1-\alpha\lambda\right)-\alpha\left(1-\lambda\right)k^n}{1-\alpha}\left|a_k\right|+\sum_{k=1}^{\infty}\frac{k^m\left(1-\alpha\lambda\right)-\left(-1\right)^{m-n}\alpha\left(1-\lambda\right)k^n}{1-\alpha}\left|b_k\right|\le a_1\,, \tag{8}$$

where $a_1 = 1+\sum_{k=2}^{\infty}a_k z_0^{k-1}-(-1)^{m-1}\sum_{k=1}^{\infty}b_k z_0^{k-1}$, $m\in N, n\in N_0, m>n$, $0\le\alpha<1$ and $0\le\lambda<1$, then f is sense-preserving, harmonic univalent in *U* and $f\in\overline{S_H}\left(m,n;\alpha;\lambda,z_0\right)$.

Proof: If $z_1\ne z_2$, then,

$$\left|\frac{f_m(z_1)-f_m(z_2)}{h(z_1)-h(z_2)}\right|\ge 1-\left|\frac{g_m(z_1)-g_m(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty}b_k(z_1^k-z_2^k)}{a_1\left(z_1-z_2\right)+\sum_{k=2}^{\infty}a_k(z_1^k-z_2^k)}\right|$$

$$>1-\frac{\sum_{k=1}^{\infty}k\,|\,b_k\,|}{a_1-\sum_{k=2}^{\infty}k\,|\,a_k\,|}$$

$$\ge 1-\frac{\left(\sum_{k=1}^{\infty}\frac{k\left(1-\alpha\lambda\right)-\left(-1\right)^{m-n}\alpha\left(1-\lambda\right)}{1-\alpha}\left|b_k\right|\right)}{a_1-\sum_{k=2}^{\infty}\frac{k\left(1-\alpha\lambda\right)-\alpha\left(1-\lambda\right)}{1-\alpha}\left|a_k\right|}$$

$$\ge 1-\frac{\left(\sum_{k=1}^{\infty}\frac{k^n\left(k\left(1-\alpha\lambda\right)-\left(-1\right)^{m-n}\alpha\left(1-\lambda\right)\right)}{1-\alpha}\left|b_k\right|\right)}{a_1-\sum_{k=2}^{\infty}\frac{k^n\left(k\left(1-\alpha\lambda\right)-\alpha\left(1-\lambda\right)\right)}{1-\alpha}\left|a_k\right|}$$

$$\ge 1-\frac{\left(\sum_{k=1}^{\infty}\frac{k^{n+1}\left(1-\alpha\lambda\right)-\left(-1\right)^{m-n}\alpha\left(1-\lambda\right)k^n}{1-\alpha}\left|b_k\right|\right)}{a_1-\sum_{k=2}^{\infty}\frac{k^{n+1}\left(1-\alpha\lambda\right)-\alpha\left(1-\lambda\right)k^n}{1-\alpha}\left|a_k\right|}$$

$$\geq 1-\frac{\left(\sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}|b_k|\right)}{a_1-\sum_{k=2}^{\infty}\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}|a_k|},$$

since $(m>n)$

$$\geq 0, \qquad \text{(Using (8))}$$

which proves univalence.

Also we have

$$|h'(z)|\geq a_1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1}$$
$$>a_1-\sum_{k=2}^{\infty}k|a_k|$$
$$\geq a_1-\sum_{k=2}^{\infty}\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}|a_k|$$
$$\geq \sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}|b_k|$$
$$>\sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}|b_k||z|^{k-1}$$
$$\geq \sum_{k=1}^{\infty}k|b_k||z|^{k-1}$$
$$\geq |g'(z)|.$$

Hence *f* is sense preserving in *U*.

Using the fact that $\operatorname{Re}\omega>\alpha$ if and only if

$|1-\alpha+\omega|>|1+\alpha-\omega|$, it suffices to show that

$$\left|(1-\alpha)\left\{\lambda D^m f(z)+(1-\lambda)D^n f(z)\right\}+D^m f(z)\right|-\left|(1+\alpha)\left\{\lambda D^m f(z)+(1-\lambda)D^n f(z)\right\}-D^m f(z)\right|>0. \quad (9)$$

Substituting for $D^m f(z)$ and $D^n f(z)$ in L.H.S. of (9), we have

$$=\left|(2-\alpha)a_1z+\sum_{k=2}^{\infty}\left[(1-\alpha)\left(\lambda k^m+(1-\lambda)k^n\right)+k^m\right]a_kz^k+(-1)^n\sum_{k=1}^{\infty}\left[(-1)^{m-n}(1-\alpha)\lambda k^m+(1-\alpha)(1-\lambda)k^n+(-1)^{m-n}k^m\right]b_k\bar{z}^k\right|$$
$$-\left|\alpha a_1z+\sum_{k=2}^{\infty}\left[(1+\alpha)\left\{\lambda k^m+(1-\lambda)k^n\right\}-k^m\right]a_kz^k+(-1)^n\sum_{k=1}^{\infty}\left[(-1)^{m-n}\lambda k^m(1+\alpha)+(1+\alpha)(1-\lambda)k^n-(-1)^{m-n}k^m\right]b_k\bar{z}^k\right|$$

$$\geq 2(1-\alpha)a_1|z|-\sum_{k=2}^{\infty}2\left[k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n\right]|a_k||z|^k-\sum_{k=1}^{\infty}\left|(-1)^{m-n}\left[(1-\alpha)\lambda k^m+k^m\right]+(1-\alpha)(1-\lambda)k^n\right||b_k||z|^k$$
$$-\sum_{k=1}^{\infty}\left|(-1)^{m-n}\left[(1+\alpha)\lambda k^m-k^m\right]+(1+\alpha)(1-\lambda)k^n\right||b_k||z|^k$$

$$=\begin{cases} 2(1-\alpha)a_1|z|-2\sum_{k=2}^{\infty}\left[k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n\right]|a_k||z|^k-2\sum_{k=1}^{\infty}\left[k^m(1-\alpha\lambda)+\alpha(1-\lambda)k^n\right]|b_k||z|^k \; if\ m-n\, is\, odd \\ 2(1-\alpha)a_1|z|-2\sum_{k=2}^{\infty}\left[k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n\right]|a_k||z|^k-2\sum_{k=1}^{\infty}\left[k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n\right]|b_k||z|^k \; if\ m-n\, is\, even \end{cases}$$

$$=2(1-\alpha)|z|\left\{a_1-\sum_{k=2}^{\infty}\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}|a_k||z|^{k-1}-\sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}|b_k||z|^{k-1}\right\}$$

$$\geq 2(1-\alpha)\left\{a_1-\sum_{k=2}^{\infty}\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}|a_k|-\sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}|b_k|\right\}.$$

The last expression is non negative by (8), and so the direct part of the theorem is proved.

Conversely, for functions f_m of the form (7), we notice that the condition

$$\mathrm{Re}\left\{\frac{D^m f_m(z)}{\lambda D^m f_m(z)+(1-\lambda)D^n f_m(z)}\right\}>\alpha$$

is equivalent to

$$\mathrm{Re}\left\{\frac{(1-\alpha)a_1 z-\sum_{k=2}^{\infty}\left[k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n\right]a_k z^k+(-1)^{2m-1}\sum_{k=1}^{\infty}\left[k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n\right]b_k\overline{z}^k}{a_1 z-\sum_{k=2}^{\infty}\left[\lambda k^m+(1-\lambda)k^n\right]a_k z^k+(-1)^{2m-1}\sum_{k=1}^{\infty}\left[\lambda k^m+(-1)^{m-n}(1-\lambda)k^n\right]b_k\overline{z}^k}\right\}\geq 0. \quad (10)$$

The above required condition (10) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\frac{a_1(1-\alpha)-\sum_{k=2}^{\infty}\left[k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n\right]a_k r^{k-1}-\sum_{k=1}^{\infty}\left[k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n\right]b_k r^{k-1}}{a_1-\sum_{k=2}^{\infty}\left[\lambda k^m+(1-\lambda)k^n\right]a_k r^{k-1}-\sum_{k=1}^{\infty}\left[\lambda k^m+(-1)^{m-n}(1-\lambda)k^n\right]b_k r^{k-1}}\geq 0. \quad (11)$$

If the condition (8) does not hold, then the numerator in (11) is negative for r sufficiently close to 1.

Hence there exist $z_0=r_0$ in (0, 1) for which the quotient in (11) is negative. This contradicts the required condition for $f\in\overline{S_H}(m,n;\alpha;\lambda,z_0)$ and so the proof is complete.

Next we determine the extreme points of closed convex hulls of $\overline{S_H}(m,n;\alpha;\lambda,z_0)$ denoted by clco $\overline{S_H}(m,n;\alpha;\lambda,z_0)$.

Theorem 2.2: Let f_m be given by (7). Then $f_m\in\overline{S_H}(m,n;\alpha;\lambda,z_0)$ if and only if

$f_m(z)=\sum_{k=1}^{\infty}\left(x_k h_k(z)+y_k g_{mk}(z)\right)$, where $h_1(z)=z$,

$h_k(z)=z-\frac{1-\alpha}{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}z^k \ (k=2,3,4....)$,

$$g_{mk}(z)=z+(-1)^{m-1}\frac{1-\alpha}{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}\bar{z}^k,\ (k=1,2,3....)\ ,\ x_k\geq 0,\ y_k\geq 0,\ \sum_{k=1}^{\infty}(x_k+y_k)=a_1.$$

In particular the extreme points of $\overline{S_H}(m,n;\alpha;\lambda,z_0)$ are $\{h_k\}$ and $\{g_{mk}\}$.

Proof: Suppose

$$f_m(z)=\sum_{k=1}^{\infty}\left(x_k h_k(z)+y_k g_{mk}(z)\right)$$

$$=\sum_{k=1}^{\infty}(x_k+y_k)z-\sum_{k=2}^{\infty}\frac{1-\alpha}{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}x_k z^k+(-1)^{m-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}y_k\bar{z}^k.$$

Then $\displaystyle\sum_{k=2}^{\infty}\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}\left(\frac{1-\alpha}{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}x_k\right)$

$$+\sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\left(\frac{1-\alpha}{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}y_k\right)$$

$$=\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k$$

$$=a_1-x_1\leq a_1,$$

and so $f_m\in\overline{S_H}(m,n;\alpha;\lambda,z_0)$.

Conversely, if $f_m\in clco\overline{S}_H(m,n;\alpha;\lambda,z_0)$ then

$$a_k\leq\frac{1-\alpha}{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n},\ (k=2,3,4....)\ \text{and}\ b_k\leq\frac{1-\alpha}{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n},\ (k=1,2,3....).$$

Set $x_k=\dfrac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}a_k$, $(k=2,3,4....)$ and $y_k=\dfrac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}b_k$, $(k=1,2,3....)$. Then by Theorem 2.1, $0\leq x_k\leq a_1$, $(k=2,3,4....)$ and $0\leq y_k\leq a_1$, $(k=1,2,3....)$. We define $x_1=a_1-\sum_{k=2}^{\infty}x_k-\sum_{k=1}^{\infty}y_k$ and note that by Theorem 2.1 $x_1\geq 0$. Consequently, we obtain $f_m(z)=\sum_{k=1}^{\infty}\left(x_k h_k(z)+y_k g_{mk}(z)\right)$ as required.

The following theorem gives the bounds for functions in $\overline{S_H}(m,n;\alpha;\lambda,z_0)$ which yields a covering result for this class.

Theorem 2.3: Let $f_m\in\overline{S_H}(m,n;\alpha;\lambda,z_0)$. Then for $|z|=r<1$ we have

$$|f_m(z)|\leq(a_1+b_1)r+\frac{1}{2^n}\left(\frac{1-\alpha}{2^{m-n}(1-\alpha\lambda)-\alpha(1-\lambda)}-\frac{(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)}{2^{m-n}(1-\alpha\lambda)-\alpha(1-\lambda)}b_1\right)r^2,|z|=r<1$$

and

$$|f_m(z)|\geq(a_1-b_1)r-\frac{1}{2^n}\left(\frac{1-\alpha}{2^{m-n}(1-\alpha\lambda)-\alpha(1-\lambda)}-\frac{(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)}{2^{m-n}(1-\alpha\lambda)-\alpha(1-\lambda)}b_1\right)r^2,|z|=r<1.$$

Proof: We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted.

Let $f_m \in \overline{S_H}(m,n;\alpha;\lambda,z_0)$. Taking the absolute value of f_m we have

$$\begin{aligned}|f_m(z)| &\le (a_1+b_1)r + \sum_{k=2}^{\infty}(a_k+b_k)r^k\\
&\le (a_1+b_1)r + \sum_{k=2}^{\infty}(a_k+b_k)r^2\\
&= (a_1+b_1)r + \frac{1-\alpha}{2^m(1-\alpha\lambda)-2^n\alpha(1-\lambda)}\sum_{k=2}^{\infty}\frac{2^m(1-\alpha\lambda)-2^n\alpha(1-\lambda)}{1-\alpha}(a_k+b_k)r^2\\
&\le (a_1+b_1)r + \frac{(1-\alpha)r^2}{2^m(1-\alpha\lambda)-2^n\alpha(1-\lambda)}\left(\sum_{k=2}^{\infty}\left(\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}a_k+\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}b_k\right)\right)\\
&\le (a_1+b_1)r - \frac{1}{2^n}\left(\frac{1-\alpha}{2^{m-n}(1-\alpha\lambda)-\alpha(1-\lambda)}-\frac{(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)}{2^{m-n}(1-\alpha\lambda)-\alpha(1-\lambda)}b_1\right)r^2.\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 2.3.

Corollary 2.4: Let f_m of the form (4) be so that $f_m \in \overline{S_H}(m,n;\alpha;\lambda,z_0)$. Then

$$\left\{\omega : |\omega| < \frac{2^m a_1 - 1 - \alpha\left[2^m\lambda a_1 + 2^n(1-\lambda)a_1 - 1\right]}{2^m(1-\alpha\lambda)-\alpha(1-\lambda)2^n} - \frac{2^m - 1 - \alpha\left[2^m\lambda + 2^n(1-\lambda) - \lambda - (-1)^{m-n}(1-\lambda)\right]}{2^m(1-\alpha\lambda)-\alpha(1-\lambda)2^n}b_1\right\}.$$

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form $f_m(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k \bar{z}^k$ and $F_m(z) = a_1 z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} B_k \bar{z}^k$ we define the convolution of two harmonic functions f and F as

$$(f_m * F_m)(z) = f_m(z) * F_m(z) = a_1 z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k B_k \bar{z}^k. \qquad (12).$$

Using this definition, we show that the class $\overline{S_H}(m,n;\alpha;\lambda,z_0)$ is closed under convolution.

Theorem 2.5: For $0 \le \beta \le \alpha < 1$ let $f_m \in \overline{S_H}(m,n;\alpha;\lambda,z_0)$ and $F_m \in \overline{S_H}(m,n;\beta;\lambda,z_0)$. Then $f_m * F_m \in \overline{S}_H(m,n;\alpha;\lambda,z_0) \subseteq \overline{S}_H(m,n;\beta;\lambda,z_0)$.

Proof: Let $f_m(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k \bar{z}^k$ be in $\overline{S_H}(m,n;\alpha;\lambda,z_0)$ and $F_m(z) = a_1 z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} B_k z^k$ be in $\overline{S_H}(m,n;\beta;\lambda,z_0)$. Then the convolution $f_m * F_m$ is given by (12) .We wish to show that the coefficients of $f_m * F_m$ satisfy the required condition given in Theorem 2.1. For $F_m \in \overline{S}_H(m,n;\beta;\lambda,z_0)$ we note that $A_k \le 1$ and $B_k \le 1$. Now, for the convolution function $f_m * F_m$ we get

$$\sum_{k=2}^{\infty}\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}a_kA_k+\sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}b_kB_k$$
$$\le\sum_{k=2}^{\infty}\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}a_k+\sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}b_k$$
$$\le a_1. \text{ (Since } f_m\in\overline{S_H}(m,n;\alpha;\lambda,z_0)\text{)}.$$

Therefore $f_m * F_m \in \overline{S}_H(m,n;\alpha;\lambda,z_0) \subseteq \overline{S}_H(m,n;\beta;\lambda,z_0)$. for $0\le\beta\le\alpha<1$.

Next, we show that $\overline{S_H}(m,n;\alpha;\lambda,z_0)$ is closed under convex combinations of its members.

Theorem 2.6: The class $\overline{S_H}(m,n;\alpha;\lambda,z_0)$ is closed under convex combination.

Proof: For $i=1,2,3.....$ let $f_{m_i}(z)\in\overline{S_H}(m,n;\alpha;\lambda,z_0)$, where $f_{m_i}(z)$ is given by

$$f_{m_i}(z)=a_1z-\sum_{k=2}^{\infty}a_{k_i}z^k+(-1)^{m-1}\sum_{k=1}^{\infty}b_{k_i}\overline{z}^k.$$

Then by Theorem 2.1'

$$\sum_{k=2}^{\infty}\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}a_{k_i}+\sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}b_{k_i}\le a_1.$$

For $\sum_{i=1}^{\infty}t_i=1, 0\le t_i\le 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty}t_i f_{m_i}(z)=a_1z-\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_ia_{k_i}\right)z^k+(-1)^{m-1}\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}t_ib_{k_i}\right)\overline{z}^k$$

Then by (8),

$$\sum_{k=2}^{\infty}\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}\sum_{i=1}^{\infty}t_ia_{k_i}+\sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\sum_{i=1}^{\infty}t_ib_{k_i}$$
$$=\sum_{i=1}^{\infty}t_i\left\{\sum_{k=2}^{\infty}\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}a_{k_i}+\sum_{k=1}^{\infty}\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}b_{k_i}\right\}$$
$$\le a_1\sum_{i=1}^{\infty}t_i=a_1.$$

This is the condition required by Theorem 2.1 and so $\sum_{i=1}^{\infty}t_i f_{m_i}(z)\in\overline{S_H}(m,n;\alpha;\lambda,z_0)$.

Theorem 2.7: If $f_m\in\overline{S}_H(m,n;\alpha;\lambda,z_0)$ then f_m is convex in the disc

$$|z|\le\min_k\left\{\frac{(1-\alpha)(a_1-b_1)}{k\left[(1-\alpha)-\left\{(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)\right\}b_1\right]}\right\}^{1/k-1},\quad (k=2,3,4....).$$

Proof: Let $f_m \in \overline{S}_H(m,n;\alpha;\lambda,z_0)$, and let $r(0<r<1)$ be fixed. Then $r^{-1}f_m(rz) \in \overline{S}_H(m,n;\alpha;\lambda,z_0)$ and we have

$$\begin{aligned}\sum_{k=2}^{\infty} k^2(a_k+b_k)r^{k-1} &= \sum_{k=2}^{\infty} k(a_k+b_k)(kr^{k-1})\\ &\le \sum_{k=2}^{\infty}\left(\frac{k^m(1-\alpha\lambda)-\alpha(1-\lambda)k^n}{1-\alpha}a_k+\frac{k^m(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}b_k\right)kr^{k-1}\\ &\le \left[a_1-\left\{\frac{(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)}{1-\alpha}\right\}b_1\right]kr^{k-1}\\ &\le a_1-b_1,\end{aligned}$$

provided

$$kr^{k-1} \le \frac{a_1-b_1}{a_1-\left\{\dfrac{(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)}{1-\alpha}b_1\right\}}$$

which is true if

$$r \le \min_k \left\{\frac{(1-\alpha)(a_1-b_1)}{k\left[a_1(1-\alpha)-\left\{(1-\alpha\lambda)-(-1)^{m-n}\alpha(1-\lambda)\right\}b_1\right]}\right\}^{1/(k-1)} \quad (k=2,3,4....)$$

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