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CHARACTERIZATIONS OF 2-QUASI TOTAL COLORING GRAPHS

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ABSTRACT

This paper commences with new discernment of 2-quasi total colouring. It establishes various results on 2-quasi total colouring. By using new proof technique of H.P. yap it has been prove that the total colouring conjecture is extended for 2- quasi total graph of any graph G of order n having maximum degree at least n-3.

Key words: *2-quasi total graphs, complete bipartite graph, matching, saturation.*

AMS subject classification*: 05C15, 05C30, 05C99.*

0. INTRODUCTION

Behzad, Chartand and Cooper (1967) proved the total coloring Conjecture for complete graphs. Rosenfeld and Vijayaditya (1971)proved this conjecture for graphs G having $\Delta(G) \leq 3$ Also the same conjecture was proved for graphs G having $\Delta(G) = 4$ by Kostochka (1977)and for complete 3-paratite graphs ,complete balanced r-partite graphs by Rosenfeld(1971) .Also the conjecture proved for complete r-partite graphs by Yap(1989).A survey on total colourings of graphs is given by Behzad (1987). The concept of quasi total graphs introduced by Basavanagoud [1998] and the 2-quasi total colouring was discussed by srinivasulu [2006].Also various properties of 1-quasi total graphs and bounds of 1-quasi total colouring was established by R. V. N. S. Rao and others (2012).

In this paper, all the graphs are considered to be finite, simple and undirected. Let G be a graph. We denote its vertex set, edge set, Chromatic index and the Maximum degree of its vertices by V(G),E(G), $\chi'(G)$ and $\Delta(G)$ respectively. We denote N(v) the neighborhood of v and d(v) is degree of vertex v. If F $\subseteq$E(G) , then G-F is the graph obtained from G by deleting F from G. If S$\subseteq$V (G), then G[S] and G-S denote the sub graphs of G induced by S and V (G)-S respectively. O_m is the null graph of order m .Other terms and notations in this paper can be found in, some topics in graph theory by H.P. yap, Cambridge Univ. Press, 1986.

1. PRELIMINARIES

1.1 ***Definition***:[12]. Let G be a graph with vertex set V (G) and edge set E (G). The *2-quasitotal graph* of G, denoted by Q_2 (G) is defined as follows: The vertex set of Q_2(G), that is, V(Q_2(G)) = V(G) $\cup$ E(G). Two vertices x, y in V (Q_2 (G)) are adjacent in Q_2 (G) if it satisfies one of the following conditions.

(i) x, y are in V(G) and $\overline{xy} \in$ E(G).
(ii) x is in V(G); y is in E(G); and x, y are incident in G.

1.2 ***Note***: (i) G is a sub graph of Q_2 (G);

1.3 ***Example***: Consider the graph G given in Fig 1.1

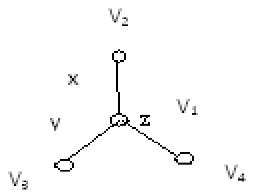

Figure 1.1: Graph with 4vertices

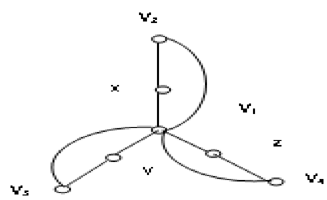

Figure 1.2: 2-Quasi total graph of figure1.1

Let us construct the graph $Q_2(G)$ of G.

$V(Q_2(G)) = V(G) \cup E(G).$
$= \{v_1, v_2, v_3, v_4\} \cup \{x, y, z\}$
$= \{v_1, v_2, v_3, v_4, x, y, z\}.$

$E(Q_2(G)) = \{ \overline{v_1v_2}, \overline{v_1v_3}, \overline{v_1v_4}, \overline{xv_1}, \overline{xv_2}, \overline{yv_1}, \overline{yv_3}, \overline{zv_1}, \overline{zv_4} \}.$

The $Q_2(G)$ is given in 1.2

1.4. Example: Fig 1.4 is 2-quasi total graph of fig 1.3

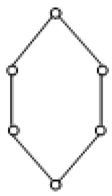

Figure1.3: A graph of order 6

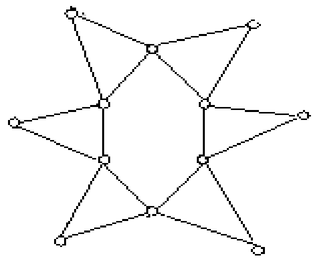

Figure 1.4: 2-Quasi total graph of figure1.3

1.5. Note: Here we briefly explain the new proof technique of H.P. Yap (1989) .That is by adding a new vertex to a graph and creating a 2-quasi total colouring of the old graph from an edge coloring of the new graph

1.6. Definition: [Lovaz1986] Given a graph *G* = (*V*, *E*), a *matching M* in *G* is a set of pair wise non-adjacent edges. That is, no two edges share a common vertex.

1.7. Example: in the graph G of figure 1.5 the sets $M_1 = \{e_1, e_2\}$ and $M_2 = \{e_1, e_3, e_4\}$ are both matchings.

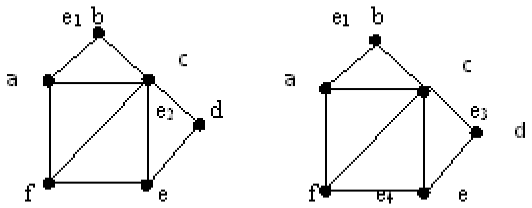

Figure 1.5: A graph G with two matchings

1.8. Definition [3]: A vertex is called *saturated* if it is an endpoint of one of the edges in the matching. Otherwise the vertex is unsaturated.

1.9. Example: From the figure 1.5, the vertices a, b, c and e are all M_1-saturated while the vertices f and d are both M_1-unsaturated and every vertex of G is M_2-saturated.

1.10. Definition [3]: A matching *M* of a graph *G* is *maximal* if every edge in *G* has a non-empty intersection with at least one edge in *M*. The following figure 1.6 shows examples of maximal matchings (doted lines in red colour) in three graphs.

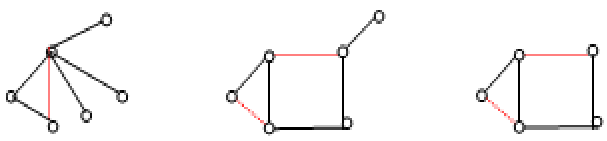

Figure 1.6 Graphs with their Maximal Matchings in doted lines

2. VARIOUS PROPERTIES OF 2-QUASI TOTAL GRAPHS

2.1 Lemma: If e = $\overline{uv} \in$ E(G), then there exist a triangle in E(Q_2(G)) containing e as one of the edges.

Proof: Suppose G is a graph with |E (G)| = 1.

Let e $\in$ E (G) and e = $\overline{vu}$ for some v, u $\in$ V(G).

Now e, v, u $\in$ V (G) $\cup$ E (G) = V (Q_2 (G))

Now $\overline{vu} \in$ E (G) $\subseteq$ E (Q_2 (G)).

Since e and u are incident in G, we have that $\overline{ue} \in$ E (Q_2 (G))

Since e and v are incident in G, we have that $\overline{ev} \in$ E (Q_2 (G)).

So $\overline{vu}$, $\overline{ue}$, $\overline{ev} \in$ E(Q_2(G)) and these edges put together form a triangle.

The proof is complete.

2.2 Theorem: For any graph with v>1 and $e \geq 1$, the Q_2 (G) has at least one 3-cycle.

Proof: Since every edge in G has two end vertices and every vertex in the edge is adjacent to at least one vertex in G. By the definition of Q_2(G) it has at least one 3- cycle.

2.3.Theorem: For any $K_{m,n}$ (m $\neq$ n) graph, the degree of the vertices in X- set is 2n, the degree of the vertices in Y- set is 2m the remain vertices have degree '2' in $Q_2(K_{m,n})$ of $K_{m,n}$.

Proof: Let X, Y be partition of $K_{m,n}$, X contain m vertices and Y contain n vertices.

Let v $\in$ X, then v is adjacent with n vertices in Y and v is incident with n edges.

Therefore v has 2n degree in $Q_2(K_{m,n})$. Hence degree of the vertices in X is of degree 2n in $Q_2(K_{m,n})$. Similarly let u$\in$Y, then u is adjacent with m vertices in X and u is adjacent with m edges. Therefore u has 2m degree in $Q_2(K_{m,n})$. Hence degree of the vertices Y is of degree 2m.

Since 'e' is incident with two vertices v_1, v_2 where e is an edge in $K_{m,n}$ .Therefore degree of 'e' in $Q_2(K_{m,n})$ is '2'

3. 2-QUASI –TOTAL CHROMATIC NUMBER

3.1 Definition: A *2-Quasi total colouring* of a graph G is an assignment of colours to the vertices and edges of G such that distinct colours are assigned to

(i) Every two adjacent vertices
(ii) Every incident vertex and edge.

A 2-quasi k-total coloring of a graph G is a quasi -total coloring of G from a set of k-colors of G. The 2-quasi total chromatic number of a graph G is the minimum positive integer k for which G is k-quasi total colorable denoted by $\chi''_{Q_2}(G)$ or $\chi(Q_2(G))$.

3.2. Example: From the figure 1.4 we say that the 2-quasi total chromatic number of fig 1.3 is 3

3.3. Theorem: The 2-quasi chromatic number of $\chi''_{Q_2}(G) \geq 3$.

Proof: By lemma 2.1, if e is an edge in G, then there exists a triangle in $E(Q_2(G))$ Containing E as one of the edges.

Let e_i be an edge of G,

Let v_i and v_j be the vertices incident with e_i

Therefore the vertex e_i in Q_2 (G) is adjacent with vertices v_i and v_j in Q_2 (G) (Since by the theorem 2.3)

Therefore there exists at least on 3-cycle in Q_2 (G)

Hence $\chi(Q_2(G)) \geq 3$.

3.4. Corollary: - If G is any graph then lower bound for its chromatic number of 2-quasi total graph is three.

3.5. Remark: From the definition of total chromatic number it is clear that $\chi''(G) \geq \Delta + 1$.But this result may not true for 2 quasi –total graphs. This can be justified by the following example.

The fig 1.2 is the 2-quasi total graph of fig 1.1.From fig 1.1 Δ=3 and from fig 1.2 $\chi''_{Q_2}(G) = 3$. But Δ+1=4. Hence the 2-quasi total chromatic number is $\chi''_{Q_2}(G) \geq\neq \Delta + 1$.

3.6. Lemma: [Behzad (1964) and Vizing (1965)] Total colouring conjecture as for any graph G, $\chi''(G) \leq \Delta + 2$.Now we have to extend this conjecture for 2-quasi total graphs.

3.7. Theorem: For any graph G of order n having Δ(G) =n-3, $\chi''_{Q_2}(G) \leq n - 1$.

To prove this theorem we shall apply the following results:

3.8. Theorem [Rosenfield, Vijaditya and Kostochka (1971)].For any graph G having $\Delta(G) \leq 4$, $\chi''(G) \leq \Delta + 2$.

3.9. Theorem Vizing (1965)]: For any graph G having at most two vertices of maximum degree then $\chi'(G) = \Delta$.

3.10. Theorem. [Cheywynd and Hilton].Let G be a connected graph of order n with three vertices of maximum degree Then $\chi'(G) = \Delta + 1$ if and only if G has three vertices of degree n-1 and the remaining vertices have degree n-2.

3.11. Lemma: For any sub graph H of G $\chi''_{Q_2}(H) \subseteq \chi''_{Q_2}(G)$.

It is trivially true for any sub graph

3.12. Lemma: Let G be a graph of order n and let maximum degree of G be Δ .Suppose there exists S $\subseteq$ V(G) such that G[S] = 0_r where r=n-Δ-1.If G-s contains a matching M such that the graph G^* obtained by adding a new vertex $c^* \notin$ V(G) to G-M and adding an edge joining to c^* to each vertex in G-S has chromatic index Δ+1,then $\chi''_{Q_2}(G) \leq \Delta + 2$.

Proof: We first note that $\Delta(G^*) = \Delta+1$.

Let π be a proper edge colouring of G^* using colours 1, 2 … Δ+1.Then we can turn π in to a 2-quasi total colouring ϕ of G using the colours 1,2,…. Δ+1, Δ+2 as follows

ϕ(v)= $\pi(c^*v)$ for any v $\in$ V(G-S)
ϕ(v)= Δ+ 2 for v $\in$ S
ϕ(e)= π(e) for any e $\in$ E (G-M); and ϕ(e) = $\pi(c^*v)$ Δ+ 2, for e $\in$ M.

We now prove our main result 3.7.

Proof of theorem 3.7: By Lemma 3.10 we can assume that G is maximal that is for any two non adjacent vertices x and y of G either d(x)=n-3.Let H = G-{x, y},V_1={z$\in$V(G)/d(z)=n-3},and M be a matching in H such that $|V(M)\cap V_1|$ is maximum among all matching in H.

We first prove that V_1 contains at most one M-Unsaturated vertex different from x and y. Suppose otherwise ,let u and v be two M-unsaturated vertices in V_1.Clearly uv$\notin$E(G).By theorem 3.8.,we can assume that $\Delta(G)\geq 5$.Thus there exists at least three vertices a_1,a_2,a_3 in H such that $a_iu\in$E(H) for i= 1,2,3.Clearly such a_i is M-saturated .Thus there exist distinct vertices b_1,b_2,b_3 in H such that a_1b_1,a_2b_2,a_3b_3 $\in$M. If $b_iv\in$E(H) for some i=1,2,3,then $M'=(M-\{a_ib_i\})\cup\{a_iu,b_iv\}$ is a matching in H such that $|V(M')\cap V_1|>|V(M)\cap V_1|$, a contradiction to our assumption .hence $b_iv\notin$E(H) for all i=1,2,3.This implies that d(v) $\leq$n-4which is false ,since u is also not adjacent to v in G ,d(v) $\leq$n-5.hence V_1 contains at most one M-unsaturated vertex different from x and y.

Finally let G^* be the graph obtained by adding a new vertex $c^*\notin$V(G) to G-M, and adding an edge joining c^* to each vertex in G-{x, y}.Then $\Delta(G^*)$=n-2 and G^*has at most two vertices of maximum degree n-2, namely, $c^*$and z, where z is an M-unsaturated vertex in V_1.Hence ,by theorem 3.9 $\chi'(G^*)$=n-2=$\Delta(G^*)$.

From lemma 3.11 $\chi''_{Q_2}(G)\leq\Delta+2$ where χ' (G-S) = Δ+1

But here χ' (G*) =n-2

Therefore $\chi''_{Q_2}(G)\leq n-2+1$= n-1

Hence proved.

In the example 1.3, n=4, $\chi''_{Q_2}(G)=3$ hence $\chi''_{Q_2}(G)\leq 4-1$=3.

In the example 1.4, n=6, $\chi''_{Q_2}(G)=3$ $\chi''_{Q_2}(G)\leq 6-1$=5

3.13.Remark: Since Behzad, Chartrand and Cooper(1967) proved that the total colouring conjecture is true for complete graphs ,by Lemma 3.11 it is also true for graphs of order n having Δ(G) =n-1.Furthur more ,suppose G is a graph of order n having Δ(G) =n-2.then applying the same proof technique of theorem 3.7 ,we can show that $\chi''_{Q_2}(G)\leq n$ .By Combining these facts and theorem 3.7 ,we have known that the TCC true for 2-quasi total graphs for any graph G of order n having Δ(G) $\geq$n-3.

4. CONCLUSION

In this paper we proved various properties of 2- quasi total graphs. Also we introduced 2-quasi total colouring and obtained the properties of 2-quasi total colourings. Also these results confirm that the total chromatic conjecture is true for 2-quasi total graph whose maximum degree is either very small or very big.

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