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ON THE ADDITIVE AND MULTIPLICATIVE STRUCTURE OF SEMIRINGS

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ABSTRACT

Let (S, +, •) be a semiring and (S, +, •, ≤) be a totally ordered semiring (t.o.s.r.). In this paper, we study the structure of semirings which are positive rational domains. It is established that in a PRD semiring (S, +, •), the set of additive idempotents is a completely prime multiplicative Ideal and (S,+) is a commutative semigroup if (S,+) is cancellative. We also study the structure of totally ordered semirings.

Keywords: *Non-negatively ordered; Non-positively ordered; Positively totally ordered; Negatively totally ordered; IMP.*

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INTRODUCTION

A triple (S, +, •) is called a semiring if (S, +) is a semigroup; (S, •) is a semigroup; a(b + c) = ab + ac and (b + c)a = ba + ca for every a, b, c in S. A semiring (S, +, •) is said to be totally ordered if there exists a full ordering on S under which (S, +) and (S, •) are totally ordered semigroups. A totally ordered (t.o.) semigroup (S, •) is non – negatively (non -positively) ordered if $x^2 \geq x$ ($x^2 \leq x$) for every x in S; (S, •) is positively ordered in strict sense (negatively ordered in strict sense) if $xy \geq x$ and $xy \geq y$ ($xy \leq x$ and $xy \leq y$) for every x, y in S. E[+] denotes the set of all additive idempotents. A semiring (S, +, •) is said to be a positive rational domain (PRD) if and only if (S, •) is an abelian group. Zeroid of a semiring (S, +, •) is the set of all x in S such that x + y = y or y + x = y for some y in S. We may also term this as the zeroid of (S, +). In this paper we investigate some important properties of semirings and totally ordered semirings with IMP.

Theorem 1: Let (S, +, •) be a t.o. PRD and $x \notin x+S$ and $x \notin S+x$ for every $x \in S$. Then (S, +) is positively ordered in strict sense or negatively ordered in the strict sense.

Proof: Since PRD contains multiplicative identity and using proposition 1[2], (S, +) is either non-negatively ordered or non-positively ordered.

Suppose (S, +) is non-negatively ordered

If x + y < x for some y in S, then x + 2y ≤ x + y

Also $y \geq 2y \Rightarrow x + y \geq x + 2y$

$\therefore$ x + y = x + 2y
$\therefore$ x + y = x + 2y $\in$ x + y + S, which is contradiction.

Similarly we can prove that (S, +) is negatively ordered in the strict sense if (S,+) is non-positively ordered.

Theorem 2: Let (S, +, •) be a PRD and (S, +) be cancellative. Then $|E[+]| = 0$ if $|S| > 1$. Proof: Since (S, +) is cancellative, it has at most one idempotent

i.e., additive identity

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Suppose S has the additive identity 0.

Now 0 + 0 = 0
$\Rightarrow$ (0 + 0).x = 0.x for all x $\in$ S.
$\Rightarrow$ 0.x + 0.x = 0.x
$\Rightarrow$ 0.(x + x) = 0.x
$\Rightarrow$ x + x = x (since (S, •) is cancellative)
$\Rightarrow$ x = 0
$\Rightarrow |S| = 1$, which is a contradiction.
$\therefore |E[+]| = 0$.

Theorem 3: If $|S| > 1$and (S,+, •) is a t.o. PRD in which (S,+) is cancellative, then one of the following is true.

(i) (S, +) is positively ordered in the strict sense.
(ii) (S, +) is negatively ordered in the strict sense.

Proof: Using Theorem 2, E [+] =ϕ.

Also x + x < x or x + x > x for every x in S using proposition 1[2].

Then by proposition 6[1], (i) and (ii) are the two possibilities.

Theorem 4: Let $a \in A^+$ in a t.o. PRD. Then the following are true.

(i) If a$\in$E [+], then a is a maximal element as well as multiplicative zero.
(ii) If S contains a maximal element 0, then na = 0 if (S,+) is non-negatively ordered.

Proof:
(i) If possible let a < y. since $a \in A^+$, there exists a natural number n such that na $\geq$ y.

Since a $\in$ E [+], a = na $\geq$ y, which is a contradiction. Thus a is a maximal element. Since A^+ and E [+] are multiplicative ideals, ax and xa are maximal elements.

Thus a = ax = xa for every x in S.

(ii)If a < 0, then na $\geq$ 0 for some natural number n, since (S, +) is non-negatively ordered. But 0 $\geq$ na as 0 is the maximal element.

This implies na = 0.

Proposition 5: If E [+] $\neq \phi$ in a PRD (S, +, •), then E[+] is completely prime multiplicative ideal.

Proof: Let x $\in$ E [+]. Then x = x + x.

If y $\in$ S, consider xy = (x + x) y = xy + xy

$\therefore$ xy $\in$ E [+]

Similiarly, yx $\in$ E [+]

$\therefore$ E[+] is multiplicative ideal.

To show that E[+] is completely prime ideal.

Let xy $\in$ E [+]. Now, xy = xy + xy = (x + x) y $\Rightarrow$ x = x + x, since (S, •) is a group, cancellation laws holds good.
$\Rightarrow$ x $\in$ E [+]

Also xy = xy + xy = x (y + y) $\Rightarrow$ y = y + y (using left cancellation law)

$\Rightarrow$ y = y + y $\Rightarrow$ y $\in$ E[+] either x $\in$ E[+] or y $\in$ E[+].

Theorem 6: Let (S, +, •) be a PRD. Then (S, +) is a band if E [+] $\neq \phi$. In particular if S contains an additive identity.

Proof: If $x \in E[+]$, then for any y in S
$xy = (x + x)\, y = xy + xy = x\,(y + y) \Rightarrow y = y + y$ (Since (S, •) is a group)
$yx = y\,(x + x) = yx + yx = (y + y)\, x \Rightarrow y = y + y$

i.e., $y \in E[+]$
$\therefore S \subseteq E[+]$ .But $E[+] \subseteq S$.
$\therefore\ S = E[+]$

Since $0 \in S$ such that $0 + 0 = 0 \in E[+]$

Now $0 + 0 = 0 \Rightarrow 0.1 + 0.1 = 0.1 \Rightarrow 0(1 + 1) = 0.1 \Rightarrow 1 + 1 = 1$

(Since (S, •) is cancellative)

If $x \in S$, then $x.\,(1 + 1) = x.1 \Rightarrow x + x = x$.

Theorem 7: Let (S, +, •) be a PRD and $x \in Z$, where Z is the zeroid of (S, +, •). Then there exists an element s in S such that 1 + s = s or there exists an element $p \in S$ such that p + 1 = p for some $p \in S$.

Proof: Suppose $x \in Z$. Then $\exists\ y \in S$ such that

x + y = y or y + x = y.

Suppose   x + y = y

$\Rightarrow x^{-1}(x + y) = x^{-1}y$
$\Rightarrow x^{-1}x + x^{-1}y = x^{-1}y$
$\Rightarrow 1 + x^{-1}y = x^{-1}y$

i.e., $1 + s = s$ where $s = x^{-1}y \in S$

Suppose y + x = y

$x^{-1}(y + x) = x^{-1}y$
$\Rightarrow x^{-1}y + x^{-1}x = x^{-1}y$
$\Rightarrow x^{-1}y + 1 = x^{-1}y$

i.e., $p + 1 = p$ where $p = x^{-1}y$

Theorem 8: Let (S, +, •) be a PRD. If (S, +, •) is cancellative, then (S, +) is commutative.

Proof: Since (S, +, •) is a PRD, S contains multiplicative identity.

Now $x + y + x + y = (x + y)\,(1+1)$
$= x + x + y + y$

Since (S, +) is cancellative

y + x = x + y.

Theorem 9: Let (S, +, •) be a PRD satisfying ab = a + b + ab for all $a,b \in S$. If (S, +) is left cancellative, then (S, +) is commutative.

Proof: By hypothesis   ab = a + b + ab
ba = b + a + ba

Since (S, •) is commutative, ab = ba.

Therefore, a + b + ab = b + a + ba

Since (S, +) is left cancellative, $a + b = b + a$.

Theorem 10: If (S, +, •) is a PRD and (S, •) is rectangular band, then S reduces to a singleton set.

Proof: Suppose (S, •) is a rectangular band

$x= x(x)x$

$x = x^3$

i.e $x^2 = x^4$

Also $x = x(x^2)x$

$\Rightarrow x = x^4$

Therefore $x = x^2$

i.e (S, •) is a band.

Since (S, +, •) is a PRD (S, •) is a group. Therefore the identity is the only multiplicative idempotents in S.

Hence S reduces to a singleton set.

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