International Journal of Mathematical Archive-3(7), 2012, 2735-2738
IJMA Available online through www.ijma.info ISSN 2229 – 5046

# ON THE ADDITIVE AND MULTIPLICATIVE STRUCTURE OF SEMIRINGS

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*(Received on: 06-07-12; Accepted on: 25-07-12)*

*ABSTRACT*

*Let (S, +, •) be a semiring and (S, +, •, ≤) be a totally ordered semiring (t.o.s.r.). In this paper, we study the structure of semirings which are positive rational domains. It is established that in a PRD semiring (S, +, •), the set of additive idempotents is a completely prime multiplicative Ideal and (S,+) is a commutative semigroup if (S,+) is cancellative. We also study the structure of totally ordered semirings.*

***Keywords:*** *Non-negatively ordered; Non-positively ordered; Positively totally ordered; Negatively totally ordered; IMP.*

***2000 Mathematics Subject Classification:*** *20M10, 16Y60.*

## INTRODUCTION

A triple (S, +, •) is called a semiring if (S, +) is a semigroup; (S, •) is a semigroup; a(b + c) = ab + ac and (b + c)a = ba + ca for every a, b, c in S. A semiring (S, +, •) is said to be totally ordered if there exists a full ordering on S under which (S, +) and (S, •) are totally ordered semigroups. A totally ordered (t.o.) semigroup (S, •) is non – negatively (non -positively) ordered if $x^2 \geq x$ ($x^2 \leq x$) for every x in S; (S, •) is positively ordered in strict sense (negatively ordered in strict sense) if $xy \geq x$ and $xy \geq y$ ($xy \leq x$ and $xy \leq y$) for every x, y in S. E[+] denotes the set of all additive idempotents. A semiring (S, +, •) is said to be a positive rational domain (PRD) if and only if (S, •) is an abelian group. Zeroid of a semiring (S, +, •) is the set of all x in S such that x + y = y or y + x = y for some y in S. We may also term this as the zeroid of (S, +). In this paper we investigate some important properties of semirings and totally ordered semirings with IMP.

**Theorem 1:** Let (S, +, •) be a t.o. PRD and $x \notin x+S$ and $x \notin S+x$ for every $x \in S$. Then (S, +) is positively ordered in strict sense or negatively ordered in the strict sense.

**Proof:** Since PRD contains multiplicative identity and using proposition 1[2], (S, +) is either non-negatively ordered or non-positively ordered.

Suppose (S, +) is non-negatively ordered

If $x + y < x$ for some y in S, then $x + 2y \leq x + y$

Also $y \geq 2y \Rightarrow x + y \geq x + 2y$

$\therefore x + y = x + 2y$
$\therefore x + y = x + 2y \in x + y + S$, which is contradiction.

Similarly we can prove that (S, +) is negatively ordered in the strict sense if (S,+) is non-positively ordered.

**Theorem 2:** Let (S, +, •) be a PRD and (S, +) be cancellative. Then $|E[+]| = 0$ if $|S| > 1$. Proof: Since (S, +) is cancellative, it has at most one idempotent

i.e., additive identity

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Suppose S has the additive identity 0.

Now $0 + 0 = 0$
$\Rightarrow (0 + 0).x = 0.x$ for all $x \in S$.
$\Rightarrow 0.x + 0.x = 0.x$
$\Rightarrow 0.(x + x) = 0.x$
$\Rightarrow x + x = x$ (since (S, •) is cancellative)
$\Rightarrow x = 0$
$\Rightarrow |S| = 1$, which is a contradiction.
$\therefore |E[+]| = 0$.

**Theorem 3:** If $|S| > 1$and (S,+, •) is a t.o. PRD in which (S,+) is cancellative, then one of the following is true.

(i) (S, +) is positively ordered in the strict sense.
(ii) (S, +) is negatively ordered in the strict sense.

**Proof:** Using Theorem 2, $E[+] = \phi$.

Also $x + x < x$ or $x + x > x$ for every x in S using proposition 1[2].

Then by proposition 6[1], (i) and (ii) are the two possibilities.

**Theorem 4:** Let $a \in A^+$ in a t.o. PRD. Then the following are true.

(i) If $a \in E[+]$, then a is a maximal element as well as multiplicative zero.
(ii) If S contains a maximal element 0, then na = 0 if (S,+) is non-negatively ordered.

**Proof:**
(i) If possible let $a < y$. since $a \in A^+$, there exists a natural number n such that $na \geq y$.

Since $a \in E[+]$, $a = na \geq y$, which is a contradiction. Thus a is a maximal element. Since $A^+$ and E [+] are multiplicative ideals, ax and xa are maximal elements.

Thus a = ax = xa for every x in S.

(ii)If $a < 0$, then $na \geq 0$ for some natural number n, since (S, +) is non-negatively ordered. But $0 \geq na$ as 0 is the maximal element.

This implies na = 0.

**Proposition 5:** If $E[+] \neq \phi$ in a PRD (S, +, •), then E[+] is completely prime multiplicative ideal.

**Proof:** Let $x \in E[+]$. Then $x = x + x$.

If $y \in S$, consider $xy = (x + x) y = xy + xy$

$\therefore xy \in E[+]$

Similiarly, $yx \in E[+]$

$\therefore$ E[+] is multiplicative ideal.

To show that E[+] is completely prime ideal.

Let $xy \in E[+]$. Now, $xy = xy + xy = (x + x) y \Rightarrow x = x + x$, since (S, •) is a group, cancellation laws holds good.
$\Rightarrow x \in E[+]$

Also $xy = xy + xy = x (y + y) \Rightarrow y = y + y$ (using left cancellation law)

$\Rightarrow y = y + y \Rightarrow y \in E[+]$ either $x \in E[+]$ or $y \in E[+]$.

 

**Theorem 6:** Let (S, +, •) be a PRD. Then (S, +) is a band if $E[+] \neq \phi$. In particular if S contains an additive identity.

**Proof:** If $x \in E[+]$, then for any y in S
$xy = (x + x)\,y = xy + xy = x\,(y + y) \Rightarrow y = y + y$ (Since (S, •) is a group)
$yx = y\,(x + x) = yx + yx = (y + y)\,x \Rightarrow y = y + y$

i.e., $y \in E[+]$
$\therefore S \subseteq E[+]$ .But $E[+] \subseteq S$.
$\therefore S = E[+]$

Since $0 \in S$ such that $0 + 0 = 0 \in E[+]$

Now $0 + 0 = 0 \Rightarrow 0.1 + 0.1 = 0.1 \Rightarrow 0(1 + 1) = 0.1 \Rightarrow 1 + 1 = 1$

(Since (S, •) is cancellative)

If $x \in S$, then $x.(1 + 1) = x.1 \Rightarrow x + x = x$.

**Theorem 7:** Let (S, +, •) be a PRD and $x \in Z$, where Z is the zeroid of (S, +, •). Then there exists an element s in S such that $1 + s = s$ or there exists an element $p \in S$ such that $p + 1 = p$ for some $p \in S$.

**Proof:** Suppose $x \in Z$. Then $\exists\, y \in S$ such that

$x + y = y$ or $y + x = y$.

Suppose $x + y = y$

$\Rightarrow x^{-1}(x + y) = x^{-1}y$
$\Rightarrow x^{-1}x + x^{-1}y = x^{-1}y$
$\Rightarrow 1 + x^{-1}y = x^{-1}y$

i.e., $1 + s = s$ where $s = x^{-1}y \in S$

Suppose $y + x = y$

$x^{-1}(y + x) = x^{-1}y$
$\Rightarrow x^{-1}y + x^{-1}x = x^{-1}y$
$\Rightarrow x^{-1}y + 1 = x^{-1}y$

i.e., $p + 1 = p$ where $p = x^{-1}y$

**Theorem 8:** Let (S, +, •) be a PRD. If (S, +, •) is cancellative, then (S, +) is commutative.

**Proof:** Since (S, +, •) is a PRD, S contains multiplicative identity.

Now $x + y + x + y = (x + y)(1+1)$
$= x + x + y + y$

Since (S, +) is cancellative

$y + x = x + y$.

**Theorem 9:** Let (S, +, •) be a PRD satisfying $ab = a + b + ab$ for all $a,b \in S$. If (S, +) is left cancellative, then (S, +) is commutative.

**Proof:** By hypothesis $ab = a + b + ab$
$ba = b + a + ba$

Since (S, •) is commutative, $ab = ba$.

Therefore, $a + b + ab = b + a + ba$

 

Since (S, +) is left cancellative, a + b = b + a.

**Theorem 10:** If (S, +, •) is a PRD and (S, •) is rectangular band, then S reduces to a singleton set.

**Proof:** Suppose (S, •) is a rectangular band

$x = x(x)x$

$x = x^3$

i.e $x^2 = x^4$

Also $x = x(x^2)x$

$\Rightarrow x = x^4$

Therefore $x = x^2$

i.e (S, •) is a band.

Since (S, +, •) is a PRD (S, •) is a group. Therefore the identity is the only multiplicative idempotents in S.

Hence S reduces to a singleton set.

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**Source of support: Nil, Conflict of interest: None Declared**