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Existence results for neutral functional integrodifferential equations with delay in Banach spaces

A. Revathi[1*] & R. Thilagavathi[2]

[1]Department of Mathematics, Roever Engineering College, Elambalur, Perambalur-621 212, TN, India
[2]Department of Mathematics, Velalar College of Engineering and Technolgy, Thindal, Erode, TN, India

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ABSTRACT

In this paper, we study the existence of mild solutions for a first order neutral functional integro-differential equations in Banach spaces. The results are obtained by using Krasnoselski-Schaefer type fixed point theorem and semigroup theory.

Keywords: *Neutral differential equations, semigroup theory, fixed point.*

2010 AMS Subject Classification: *34K40, 47H10.*

1. INTRODUCTION

The theory of neutral delay differential equations has been extensively studied in the literature [1, 2, 11, 12, 13] Hernandez and Henriquez [7] obtained some existence results for neutral functional differential equations in Banach spaces, and in [8] they established the existence of periodic solutions for the same kind of equations. In both papers Hernandez and Henriquez used semigroup theory and the sadovski fixed point principle. Recently Balachandran ans sakthivel [3] studied the existence of solutions for neutral functional integrodifferential equations in Banach spaces and Dauer and Balachandran [6] discussed the nonlinear neutral integrodifferential systems with finite delay using Schaefer fixed point theorem.

On the other hand, integrodifferential equations are encountered in many areas of science, where it is necessary to take into account aftereffect or delay (for example, in control theory, biology, ecology and medicine). Especially, one always describes a model which possesses hereditary properties by integrodifferential equations in practice.

In this paper, we study the existence of solutions for first order neutral functional integrodifferential equations with finite delay of the form

$$\frac{d}{dt}[x(t) - g(t, x_t)] = Ax(t) + f\left(t, x_t, \int_0^t h(t, s, x_s)\mathrm{d}s\right),\ t \in J = [0, b],$$

$$x(t) = \phi(t), \quad t \in [-r, 0],$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{T(t),\ t \geq 0\}$, on a Banach space X, $g: J \times D \to X, h: J \times J \times D \to X$ and $f: J \times D \times X \to X$ are given functions.

Here $D = D([-r, 0], X)$ is a Banach space of all continuous functions $\phi: [-r, 0] \to X$ endowed with the norm

$\| \phi \|_D = \sup\{|\phi(\theta)|: -r \leq \theta \leq 0\}$.

Also for $x \in D([-r, b], X)$ we denote by x_t the element of D defined by $x_t(\theta) = x(t + \theta), \theta \in [-r, 0]$. Here $x_t(\cdot)$ represents the history of the time $t - r$, up to the present time t.

The rest of this paper is organized as follows: In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. The proof of our main results are given in Section 3.

Corresponding author: *A. Revathi[1*]*
[1]Department of Mathematics, Roever Engineering College, Elambalur, Perambalur, TN, India

2. PRELIMINARIES AND HYPOTHESES

Let X be a Banach space provided with norm $\|\cdot\|$. Let $A: D(A) \to X$ is the infinitesimal generator of an analytic semigroup $\{T(t),\ t \geq 0\}$, of bounded linear operators on X. If $\{T(t),\ t \geq 0\}$, is uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as closed linear operator on its domain $D(-A)^\alpha$. Further more, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\| x \|_\alpha = \| (-A)^\alpha x \|, \quad x \in D(-A)^\alpha$$

defines a norm on $D(-A)^\alpha$. For more details of fractional power of operators and semigroup theory, we refer [14]. From this theory, we define the following Lemma.

Lemma 2.1: *The following properties hold:*
1. If $0 < \beta < \alpha \leq 1$, then $X_\alpha \subset X_\beta$ and the imbedding is compact whenever the resolvent operator of A is compact.
2. For every $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that

$$\| (-A)^\alpha T(t) \| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq b.$$

Lemma 2.2: *[9] Let $v(\cdot), w(\cdot): [0, b] \to [0, \infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $\theta > 0, 0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad t \in J,$$

then

$$v(t) \leq e^{\theta^n \Gamma(\alpha)^n t^{n\alpha} / \Gamma(n\alpha)} \sum_{j=0}^{n-1} \left(\frac{\theta b^\alpha}{\alpha}\right)^j w(t),$$

for every $t \in [0, b]$ and every $n \in N$ such that $n\alpha > 1$, and $\Gamma(\cdot)$ is the Gamma function.

Definition 2.1: *A function $x: [-r, b] \to X, b > 0$ is called mild solution of the system $(1.1) - (1.2)$ if $x(t) = \phi(t)$, the restriction of $x(\cdot)$ to the interval $[0, b)$ is continuous, if for each $0 \leq t < b$ the function $AT(t-s)g(s, x_s), s \in [0, b)$, is integrable, and if the following integral equation is satisfied.*

$$x(t) = T(t)[\phi(0) - g(0, \phi)] + g(t, x_t) + \int_0^t AT(t-s) g(s, x_s) ds + \int_0^t T(t-s) f\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau\right), t \in [0, b].$$

We need the following Krasnoselski-Schaefer type fixed point theorem to prove our existence theorem.

Theorem 1: *[4] Let Φ_1, Φ_2 be two operators satisfying:*
1. Φ_1 is contraction, and

2. Φ_2 is completely continuous.

Then either

1. the operator equation $\Phi_1 x + \Phi_2 x = x$ has a solution, or

2. the set $\zeta = \{u \in X: \lambda \Phi_1(\frac{u}{\lambda}) + \lambda \Phi_2 u = u\}$ is unbounded for $\lambda \in (0,1)$.

Now we list the following hypotheses:

(H1) There exist constants $0 < \beta < 1, c_1, c_2, L_g$ such that g is X_β -valued, $(-A)^\beta g$ is continuous, and

$(i)\ \| (-A)^\beta g(t, x) \| \leq c_1 \| x \|_D + c_2, \quad (t, x) \in J \times D,$

$(ii)\ \| (-A)^\beta g(t, x_1) - (-A)^\beta g(t, x_2) \| \leq L_g \| x_1 - x_2 \|_D, (t, x_i) \in J \times D, i = 1,2.$ with $L_g\{\| (-A)^{-\beta} \| + \frac{C_{1-\beta} b^\beta}{\beta}\} < 1.$

(H2) A is the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$, of bounded linear operators on X, and $0 \in \rho(A)$ such that

$$\| T(t) \| \leq M,\ t \geq 0 \quad \text{and} \quad \| (-A)^{1-\beta} T(t) \| \leq \frac{C_{1-\beta}}{t^{1-\beta}}, \quad 0 < t \leq b,$$

for some constants $M, C_{1-\beta}$ and every $t \in J = [0, b]$.

(H3) (i) For each $(t,s) \in J \times J$, the function $h(t,s,\cdot): D \to X$ is continuous, and for each $x \in D$, the function $h(\cdot,\cdot,x): J \times J \to X$ is strongly measurable.

(ii) For each $t \in J$, the function $f(t,\cdot,\cdot): D \times X \to X$ is continuous, and for each $(x,y) \in D \times X$, the function $f(\cdot,x,y): J \to X$ is strongly measurable.

(iii) For every positive integer k there exists $\alpha_k \in L^1(0,b)$ such that

$$\sup_{\{\|x\|,\|y\|\}\leq k} \| f(t,x,y) \| \leq \alpha_k(t), \quad \text{for } t \in J a.e.$$

(iv) There exists an integrable function $m: [0,b] \to [0,\infty)$ and a constant $\alpha > 0$ such that

$$\| h(t,s,x) \| \leq \alpha m(s) \Omega_0(\| x \|_D), \quad 0 \leq s < t \leq b,\ x \in D$$

where $\Omega_0: [0,\infty) \to (0,\infty)$ is a continuous and non-decreasing function.

(H4) $\| f(t,x,y) \| \leq p(t)\Omega(\| x \|_D + \| y \|)$ for almost all $t \in J$ and all $x \in D, y \in X$, where $p \in L^1(J, R^+)$ and $\Omega: R^+ \to (0,\infty)$ is continuous and increasing with

$$\int_0^b m^*(s)\mathrm{d}s < \int_{C_0}^{\infty} \frac{ds}{\Omega(s)+\Omega_0(s)},$$

where

$$C_0 = \frac{F}{1-c_1\|(-A)^{-\beta}\|}, \quad C_2 = \frac{1}{1-c_1\|(-A)^{-\beta}\|},$$

$$B_0 = e^{C_1^n(\Gamma(\beta))^n b^{n\beta}/\Gamma(n\beta)} \sum_{j=0}^{n-1} \left(\frac{C_1 b^\beta}{\beta}\right)^j,$$

$m^*(t) = \max\{B_0 C_2 M p(t), \alpha m(t)\}$ and $F = M \| \phi \|_D \{1 + c_1 \| (-A)^{-\beta} \|\} + \{M+1\}\{c_2 \| (-A)^{-\beta} \|\} + \frac{c_2 C_{1-\beta} b^\beta}{\beta}$.

3. EXISTENCE RESULTS

In this section, we study the existence of mild solutions for the system (1.1) - (1.2).

Theorem 1: *If the assumptions* $(H1) - (H4)$ *are satisfied, then IVP (1.1)-(1.2) has at least one solution on* $[-r, b]$.

Proof: Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $\Phi: D([-r,b],X) \to D([-r,b],X)$ defined by

$$\Phi x(t) = \begin{cases} \phi(t) & \text{if} \quad t \in [-r,0], \\ T(t)\big[\phi(0) - g\big(0,\phi(0)\big)\big] + g(t,x_t) + \displaystyle\int_0^t AT(t-s)g(s,x_s)\mathrm{d}s & \\ \quad + \displaystyle\int_0^t T(t-s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s & \text{if} \quad t \in J. \end{cases}$$

From hypothesis $(H1)$ the following inequality holds.

$$\| AT(t-s)g(s,x_s) \| \leq \| (-A)^{1-\beta} T(t-s) \| \| (-A)^\beta g(s,x_s) \|$$

$$\leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}} [c_1 \| x_s \|_D + c_2]$$

Then from Bochner theorem [10], it follows that $AT(t-s)g(s,x_s)$ is integrable on $[0,t)$.

Now we decompose Φ as $\Phi = \Phi_1 + \Phi_2$ where

$$\Phi_1 x(t) = \begin{cases} 0 & \text{if } t \in [-r,0], \\ -T(t)g(0,\phi) + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)\mathrm{d}s & \text{if } t \in J. \end{cases}$$

$$\Phi_2 x(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ T(t)\phi(0) + \int_0^t T(t-s) f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s & \text{if } t \in J. \end{cases}$$

Now, we will show that the operators Φ_1 and Φ_2 satisfy all the conditions of Theorem 2.1 on $[-r,b]$.

First we show that Φ_1 is contraction on $D([-r,b],X)$. Let $x,y \in X$. From hypothesis $(H1)$, we have

$$\| \Phi_1 x(t) - \Phi_1 y(t) \| \leq \| g(t,x_t) - g(t,y_t) \| + \left\| \int_0^t AT(t-s)[g(s,x_s) - g(s,y_s)]\mathrm{d}s \right\|$$

$$\leq \| (-A)^{-\beta} \| L_g \| x_t - y_t \|_D + L_g \| x_t - y_t \|_D \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}\mathrm{d}s$$

$$\leq L_g \| x_t - y_t \|_D \left\{ \| (-A)^{-\beta} \| + \frac{C_{1-\beta} b^\beta}{\beta} \right\}$$

Taking supremum over t,

$$\| \Phi_1 x - \Phi_1 y \| \leq L_0 \| x - y \|_D,$$

where $L_0 = L_g \left\{ \| (-A)^{-\beta} \| + \frac{C_{1-\beta} b^\beta}{\beta} \right\}$. Since $L_0 < 1$, this shows that Φ_1 is contraction on $D([-r,b],X)$.

Next, we show that Φ_2 is completely continuous on $PC([-r,b],X)$. First we prove that Φ_2 maps bounded sets into bounded sets in $D([-r,b],X)$. Let B be a bounded set in $D([-r,b],X)$. Now for each $u(t) \in \Phi_2 x(t)$, then for each $t \in J, x \in B$,

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s) f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s$$

Then

$$\| u(t) \| \leq M \| \phi \|_D + M \int_0^t \left\| f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right) \right\| \mathrm{d}s$$

From hypothesis $(H3)(iii)$ we have

$$\| u(t) \| \leq M \| \phi \|_D + M \int_0^t \alpha_k(s)\mathrm{d}s.$$

for all $u \in \Phi_2(x) \subset \Phi_2(B)$. Hence $\Phi_2(B)$ is bounded.

Next, we show that Φ_2 maps bounded sets into equicontinuous sets. Let B be bounded, as above, and $h \in \Phi_2 x$ for some $x \in B$, then for each $t \in J$, we have

$$h(t) = T(t)\phi(0) + \int_0^t T(t-s) f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s$$

Let $r_1, r_2 \in J, r_1 < r_2$. Then we have

$$\begin{aligned} \| h(r_2) - h(r_1) \| \leq & \| T(r_2) - T(r_1) \| \| \phi(0) \| \\ & + \int_0^{r_1-\varepsilon} \| T(r_2-s) - T(r_1-s) \| \left\| f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right) \right\| \mathrm{d}s \\ & + \int_{r_1-\varepsilon}^{r_1} \| T(r_2-s) - T(r_1-s) \| \left\| f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right) \right\| \mathrm{d}s \\ & + \int_{r_1}^{r_2} \| T(r_2-s) \| \left\| f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right) \right\| \mathrm{d}s \end{aligned}$$

As $r_2 \to r_1$ and ε be small the right hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t), t > 0$ implies the continuity in the uniform operator topology.

The equicontinuity for the other cases $r_1 < r_2 \leq 0$ and $r_1 \leq 0 \leq r_2$ are obvious.

Next, we show that Φ_2 is continuous. Let $\{x_n\} \subset B$ and $x_n \to x$ in $D([-r,b],X)$. Then by hypothesis $(H3)(iii)$, we have

$$f\left(s, x_{n_s}, \int_0^t h(t,s,x_{n_s})\mathrm{d}s\right) \to f\left(s, x_s, \int_0^t h(t,s,x_s)\mathrm{d}s\right), \quad n \to \infty$$

and

$$\| f\left(s, x_{n_s}, \int_0^t h(t,s,x_{n_s})\mathrm{d}s\right) - f\left(s, x_s, \int_0^t h(t,s,x_s)\mathrm{d}s\right) \| \leq 2\alpha_k(s).$$

By dominated convergence theorem, we obtain the continuity of Φ_2:

$$\| \Phi_2 x_n - \Phi_2 x \| \leq \sup_{t\in J}[\| \int_0^t T(t-s)[f\left(s, x_{n_s}, \int_0^s h(s,\tau,x_{n_\tau})\mathrm{d}\tau\right) - f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s]$$

$$\to 0 \quad as n \to \infty$$

Thus Φ_2 is continuous. From the Arzela-Ascoli theorem it suffices to show that Φ_2 maps B into a precompact set in X. Let $0 < t \leq b$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $x \in B$ we define

$$(\Phi_2^\varepsilon x)(t) = T(t)\phi(0) + T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s$$

Then from the compactness of $T(t), t > 0$, the set $V_\varepsilon(t) = \{(\Phi_2^\varepsilon x)(t) : x \in B\}$ is precompact in X for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for every $x \in B$, we have

$$|(\Phi_2 x)(t) - (\Phi_2^\varepsilon x)(t)| \leq \int_{t-\varepsilon}^t \| T(t-s) \| \| f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right) \| \mathrm{d}s$$

$$\leq \int_{t-\varepsilon}^t M\alpha_k(s)\mathrm{d}s.$$

Therefore, there are precompact sets arbitrarily close to the set $V(t) = \{(\Phi_2 x)(t) : x \in B\}$. Hence the set $\Phi_2(B)$ is precompact in X. Hence the operator Φ_2 is completely continuous.

To apply the Krasnoselski-Schaefer theorem, it remains to show that the set

$$\zeta(\Phi) = \{x(\cdot) : \lambda\Phi_1(\tfrac{x}{\lambda}) + \lambda\Phi_2 x = x\}$$

is bounded for $\lambda \in (0,1)$. To this end let $x(\cdot) \in \zeta(\Phi)$. Then $\lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x$ for some $\lambda \in (0,1)$ and

$$\begin{aligned}
\| x(t) \| &= \| \lambda\Phi_1(\tfrac{x}{\lambda}) + \lambda\Phi_2 x \| \\
&= \lambda \| -T(t)g\left(0, \tfrac{x(0)}{\lambda}\right) + g\left(t, \tfrac{x_t}{\lambda}\right) + \int_0^t AT(t-s)g\left(s, \tfrac{x_s}{\lambda}\right)\mathrm{d}s \\
&\quad + T(t)\phi(0) + \int_0^t T(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s \\
&\leq M\left\|(-A)^{-\beta}\right\| [c_1\|\emptyset\|_D + c_2] + \left\|(-A)^{-\beta}\right\|[c_1\|x_t\|_D + c_2] + \frac{c_2 C_1 - \beta^{b^\beta}}{\beta} + c_1 C_{1-\beta}\int_0^t \frac{\|x_s\|_D}{(t-s)^{1-\beta}}ds + M\|\emptyset\|_D \\
&\quad + M\int_0^t p(s)\,\Omega\left(\|x_s\|_D + \int_0^s \propto m(\tau)\,\Omega_0(\|x_\tau\|_D)d\tau\right)ds.
\end{aligned}$$

$$\begin{aligned}
&\leq M\left\|(-A)^{-\beta}\right\|c_1\|\emptyset\|_D + Mc_2\left\|(-A)^{-\beta}\right\| + \left\|(-A)^{-\beta}\right\|c_1\|x_t\|_D + c_2\left\|(-A)^{-\beta}\right\| + c_1 C_{1-\beta}\int_0^t \frac{\|x_s\|_D}{(t-s)^{1-\beta}}ds + \frac{c_2 C_1 - \beta^{b^\beta}}{\beta} \\
&\quad + M\|\emptyset\|_D + M\int_0^t p(s)\,\Omega\left(\|x_s\|_D + \int_0^s \propto m(\tau)\,\Omega_0(\|x_\tau\|_D)d\tau\right)ds.
\end{aligned}$$

$$\leq F + c_1\left\|(-A)^{-\beta}\right\|\|x_t\|_D \int_0^t p(s)\,\Omega\big(\|x_s\|_D + \int_0^s \propto m(\tau)\,\Omega_0(\|x_\tau\|_D)d\tau\big)\,ds, t\epsilon\, J,$$

where

$$F = M\,\|\emptyset\|_D\left[1 + c_1\left\|(-A)^{-\beta}\right\|\right] + \{M+1\}\{\,c_2\left\|(-A)^{-\beta}\right\|\} + \frac{c_2 C_1 - \beta^{b^\beta}}{\beta}.$$

Let $\mu(t) = \max\{\,\|x(s)\|: -r \leq s \leq t\}, t\epsilon\, J$. Then $\|x_t\|_D \leq \mu(t)$ for all $t\epsilon\, J$ and there is a point $t^*\epsilon[-r,t]$ such that $\mu(t) = \|x(t^*)\|$.

Hence we have

$$\begin{aligned}\mu(t) &= \|x(t^*)\| \\ &\leq F + c_1\left\|(-A)^{-\beta}\right\|\|x_{t^*}\|_D + c_1 C_{1-\beta}\int_0^{t^*}\frac{\|x_s\|_D}{(t-s)^{1-\beta}}ds + M\int_0^{t^*} p(s)\,\Omega\left(\|x_s\|_D + \int_0^s \propto m(\tau)\,\Omega_0(\|x_\tau\|_D)d\tau\right)ds. \\ &\leq F + c_1\left\|(-A)^{-\beta}\right\|\mu(t) + c_1 C_{1-\beta}\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}}ds + M\int_0^t p(s)\Omega(\mu(s) + \alpha\int_0^s m(\tau)\Omega_0(\mu(\tau))d\tau)\,ds.\end{aligned}$$

Or

$$\begin{aligned}\mu(t) &\leq \frac{F}{1-c_1\left\|(-A)^{-\beta}\right\|} + \frac{1}{1-c_1\left\|(-A)^{-\beta}\right\|}\,c_1 C_{1-\beta}\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}}\,ds + \frac{M}{1-c_1\left\|(-A)^{-\beta}\right\|}. \\ &\qquad\{\int_0^t p(s)\Omega(\,\mu(s) + \alpha\int_0^s m(\tau)\Omega_0\big(\mu(\tau)\big)d\tau)\,ds\} \\ &\leq C_0 + C_1\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}}ds + MC_2\int_0^t p(s)\Omega(\,\mu(s) + \alpha\int_0^s m(\tau)\Omega_0\big(\mu(\tau)\big)d\tau)\,ds, t\epsilon J.\end{aligned}$$

where

$C_0 = \frac{F}{1-c_1\left\|(-A)^{-\beta}\right\|}$, $C_1 = \frac{c_1 C_{1-\beta}}{1-c_1\left\|(-A)^{-\beta}\right\|}$ and $C_2 = \frac{1}{1-c_1\left\|(-A)^{-\beta}\right\|}$.

From Lemma 2.2, we have

$$\mu(t) \leq B_0(C_0 + MC_2\int_0^t p(s)\Omega\left(\mu(s) + \alpha\int_0^s m(\tau)\Omega_0\big(\mu(\tau)\big)d\tau\right)ds)\,,$$

where

$$B_0 = e^{c_1^{n(\square(\beta)^n b^{n\beta}/\square(n\beta))}}\sum_{j=0}^{n-1}\left(\frac{C_1 b^\beta}{\beta}\right)^j.$$

Let us take the right hand side of the above inequality as $v(t)$. Then

$v(0) = B_0 C_0, \mu(t) \leq v(t), 0 \leq t \leq b$ and

$$\begin{aligned}v'(t) &\leq B_0 C_2 M p(t)\Omega\left(\mu(t) + \alpha\int_0^t m(s)\Omega_0\big(\mu(s)\big)ds\right) \\ &\leq B_0 C_2 M p(t)\Omega\left(v(t) + \alpha\int_0^t m(s)\Omega_0\big(v(s)\big)ds\right)\end{aligned}$$

Let $\omega(t) = v(t) + \alpha\int_0^t m(s)\Omega_0\big(v(s)\big)ds$. Then $\omega(0) = v(0) = B_0 C_0, v(t) \leq \omega(t)$ for all $t\epsilon J$, and

$$\begin{aligned}\omega'(t) &= v'(t) + \alpha m(t)\Omega_0\big(v(t)\big) \\ &\leq B_0 C_2 M p(t)\Omega\big(\omega(t)\big) + \alpha m(t)\Omega_0\big(\omega(t)\big) \\ &\leq m^*(t)\{\,\Omega\big(\omega(t)\big) + \Omega_0\big(\omega(t)\big)\}.\end{aligned}$$

Integrating from 0 to t, we obtain

$$\int_0^t \frac{\omega'(s)}{\Omega(\omega(s)) + \Omega_0(\omega(s))} ds \leq \int_0^t m^*(s) ds$$

$$\int_{\omega(0)}^{\omega(t)} \frac{ds}{\Omega(s) + \Omega_0(s)} \leq \int_0^b m^*(s) ds < \int_{B_0 C_0}^{\infty} \frac{ds}{\Omega(s) + \Omega_0(s)}.$$

Hence there exists a constant M such that $v(t) \leq M$ for all $t\epsilon J$, and $\mu(t) \leq v(t) \leq M$ for all $t\epsilon J$.

Therefore $\|x\| = \sup\|x(t)\| = \mu(b) \leq \omega(b) \leq M$ for all $x\epsilon\, C(\varnothing)$.

This shows that the set ζ is bounded in $D([-r, b], X)$. Consequently, by Theorem 2.1, the operator Φ has a fixed point in $D([-r, b], X)$. Thus the IVP (1.1)-(1.2) has a solution on $[-r, b]$. This completes the proof.

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