International Journal of Mathematical Archive-3(7), 2012, 2680-2685
IJMA Available online through www.ijma.info ISSN 2229 – 5046

A COMMON FIXED POINT THEOREM FOR FOUR SELF MAPS IN COMPLEX VALUED AND VECTOR VALUED METRIC SPACES

[1]K. P. R Sastry, [2]G. A Naidu, [3]Tadesse Bekeshie* and [4]M. A. Rahamatulla

[1]8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam-530 017, India
[2, 3]Department of Mathematics Andhra University, Visakhapatnam-530 003, India
[4] Al-Aman College of Engineering, Visakhapatnam, India

(Received on: 25-06-12; Accepted on: 15-07-12)

ABSTRACT

In this paper we prove a common fixed point theorem for four self maps f, g, S and T in complex valued metric spaces where both {f, S} and {g, T} are weakly compatible self maps of a nonempty set X. Our result generalizes the results of Sandeep Bhatt et al [2]. We also introduce the concept of a vector valued metric space as a generalization of complex valued metric space and prove some fixed point theorems in these spaces.

AMS Mathematics Subject Classification (2010)*: 47H10, 54H25.*

Keywords*: common fixed point, weakly compatible, complex valued metric space, vector valued metric space.*

---

1. INTRODUCTION

Fixed point theory is central to many existence theorems in mathematics. One of the main tools in fixed point theory is the Banach contraction theorem (also called Banach contraction principle) which states that every contraction mapping F on a complete metric space X has a unique fixed point. There are a lot of generalizations of this theorem in the literature. These generalizations were made in two typical ways. The first method is weakening/generalizing/ the contractive condition $d(Fx, Fy) \le kd(x, y)$ while the second is allowing X to be a more general space than the metric space.

There have been a number of generalizations of metric spaces such as rectangular metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, quasimetric spaces, D-metric spaces and cone metric spaces.

Recently, A. Azam, B. Fisher and M. Khan [1], introduced the concept of complex valued metric spaces and obtained a common fixed point result for a pair of mappings satisfying a certain contraction condition. This paper was soon followed by two papers by Sandeep Bhatt, Shruti Chaukiyal and R.C. Dimiri. See [2] and [3]. The results in [2] and [3] are respectively generalizations of the corresponding results in [5] and [4] for metric spaces. In [2] the authors proved a common fixed point theorem for four self maps f, g, S and T in complex valued metric spaces where both {f, S} and {g,T} are weakly compatible self maps of a nonempty set X and satisfy the contraction condition

$$d(Sx, Ty) \le ad(fx, gy) + b\,[d(fx, Sx) + d(gy, Ty)] + c[d(fx, Ty) + d(gy, Sx)]\ .$$

In this paper we prove a common fixed point theorem for four self maps f, g, S and T in complex valued metric spaces where both {f, S} and {g, T} are weakly compatible self maps of a nonempty set X. Our result generalizes the results of Sandeep Bhatt et al [2]. We also introduce the concept of vector valued metric spaces as a generalization of complex valued metric spaces and prove some fixed point theorems in these spaces.

We first consider some notions and notations which will be needed in the sequel.

1.1 Definition [1]: Let $\mathbb{C}$ be the set of complex numbers and let $z, w \in \mathbb{C}$. Define a partial order $\le$ on $\mathbb{C}$ by $z \le w$ if $Re(z) \le Re(w)$ and $Im(z) \le Im(w)$.

Notation: We write $z < w$ if $z \le w$ and $z \ne w$ and we write $z \ll w$ if $Re(z) < Re(w)$ and $Im(z) < Im(w)$.

Corresponding author: [3]*Tadesse Bekeshie**
[2, 3]Department of Mathematics Andhra University, Visakhapatnam-530 003, India

1.2 Definition [1]: Let X be a non-empty set. Suppose that the mapping d: X×X →ℂ satisfies:

(a) $0 \leq d(x, y)$ for all x, y∈ X and d(x, y)=0 if and only if x=y;
(b) d(x, y) = d(y, x) for all x, y ∈ X;
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z ∈ X.

Then d is called a *complex valued metric* on X and (X, d) is called a *complex valued metric space* (briefly CVM space).

We extend this notion and define the notion of a vector valued metric space as follows.

1.3 Definition: We define a partial order ≤ on $\mathbb{R}^n$ as follows.

For x, y ∈ $\mathbb{R}^n$ with $x = (x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$

Define $x \leq y$ iff $x_i \leq y_i$ for each i ∈ {1,2, ..., n}.

With this partial order $\mathbb{R}^n$ is a lattice and hence the join and meet of two elements in $\mathbb{R}^n$ are meaningful. For two elements x, y of $\mathbb{R}^n$, we write $x \ll y$ if $x_i < y_i$ for each i ∈ {1, 2, ..., n}. Similarly $x < y$ means $x_i < y_i$ for some i ∈ {1, 2, ..., n}. More over the join of two elements $x=(x_1,\ldots,x_2)$ and $y=(y_1,\ldots,y_2)$ of $\mathbb{R}^n$ satisfies the following properties.

Property:

i) max {x, y} = (max $\{x_1,y_1\}$, … , max $\{x_n, y_n\}$)
ii) Let $\{x^{(m)}\}$, $\{y^{(m)}\}$ be two sequences in $\mathbb{R}^n$ such that $x^{(m)} \to x$ and $y^{(m)} \to y$.

Then max $\{x^{(m)}, y^{(m)}\} \to \max\{x, y\}$ (the convergence being the norm convergence in $\mathbb{R}^n$)

1.4 Definition: Let X be a non-empty set. Suppose that the mapping d: X×X→ $\mathbb{R}^n$ satisfies:

(a) $0 \leq d(x, y)$ for all x, y ∈ X and d(x, y) =0 if and only if x=y;
(b) d(x, y) = d(y, x) for all x, y ∈ X;
(c) $d(x, y) \leq d(x, z)+d(z, y)$ for all x, y, z ∈ X.

Then d is called an *n dimensional vector valued metric* on X and (X, d) is called an *n dimensional vector valued metric space* (briefly n-dimensional VVM space).

A vector valued metric induces a Hausdorff topology τ on X as follows. Let $0 \ll r$, r∈ $\mathbb{R}^n$. Define B(x, r) :={ y ∈X: $d(x, y) \ll r$}. The family F = {B(x, r): x ∈ X, $0 \ll r$∈ $\mathbb{R}^n$} is a subbase for a topology τ on X and this topology is Hausdorff. We can then define other topological notions (like open set, closed set, interior point, limit point, etc) on X in the usual manner. See [1].

1.5 Definition: Let $\{x_m\}$ be a sequence in a VVM space (X, d), x∈ X and $\mathbf{0} \ll r$, r ∈ $\mathbb{R}^n$. We say that

a) x_m *converges* to x (or equivalently x is the limit of x_m), written $x_m \to x$, if for every r∈ $\mathbb{R}^n$ with $0 \ll r$ there is a positive integer N such that for all m>N, $d(x_m, x) \ll r$.
b) x_m is a *Cauchy sequence* if for every r∈ $\mathbb{R}^n$ with $0 \ll r$ there is a positive integer N such that for all m, k >N, $d(x_m, x_k) \ll r$.
c) (X, d) is a *complete VVM space* if every Cauchy sequence in (X, d) is convergent to an element in (X, d).

The proofs of the following lemmas for the case n=2 can be found in [2]. The case $n \geq 3$ can be proved in the same way, the norm ||.|| being the Euclidean norm in $\mathbb{R}^n$.

1.6 Lemma: Let (X, d) be a vector valued metric space and $\{x_m\}$ a sequence in X. Then $\{x_m\}$ converges to x if and only if $\| d(x_m, x)\| \to 0$ as $m \to \infty$.

1.7 Lemma: Let (X, d) be a vector valued metric space and $\{x_m\}$ a sequence in X. Then $\{x_m\}$ is a Cauchy sequence if and only if $\|d(x_m, x_{k+m})\| \to 0$ as $m \to \infty$.

1.8 Definition: Let f and g be self-maps on a set X, if w =fx = gx for some x in X, then x is called a *coincidence point* of f and g, and w is called a *point of coincidence* of f and g.

1.9 Definition [6]: Let f and g be two self-maps defined on a set X. Then f and g are said to be *weakly compatible* if they commute at their coincidence points.

2. MAIN RESULTS

Sandeep Bhatt et al [2] proved the following theorem. The corresponding theorem in the setting of metric sapces is proved by Hardy and Rogers [4].

2.1 Theorem [2]: Let (X, d) be a complex valued metric space and let f, g, S and T be four self-maps of X such that $T(X) \subseteq f(X)$ and $S(X) \subseteq g(X)$. Suppose there exist nonnegative real numbers a, b, and c with a + 2b +2c < 1 such that

$$d(Sx, Ty) \le ad(fx, gy)+b[d(fx, Sx) + d(gy, Ty)] + c[d(fx, Ty) + d(gy, Sx)] \quad (2.1.1)$$

Suppose that the pairs {f, S} and {g, T} are weakly compatible. Then f, g, S and T have a unique common fixed point.

This theorem is proved in [2] tacitly assuming that X is complete.

Now we state our main theorem. The corresponding theorem in the setting of metric spaces is given in [7] or it can be obtained as a consequence of zamifrescu's fixed point theorem [8].

2.2 Theorem: Let (X, d) be a complete complex valued metric space and let f, g, S and T are four self-maps of X such that $T(X) \subseteq f(X)$ and $S(X) \subseteq g(X)$ and satisfying

$$d(Sx, Ty) \le \lambda \max \left\{ d(fx, gy), \frac{d(fx,Sx)+d(gy,Ty)}{2}, \frac{d(fx,Ty)+d(gy,Sx)}{2} \right\} \quad (2.2.1)$$

Suppose that the pairs {f, S} and {g, T} are weakly compatible and T(X) is closed. Then f, g, S and T have a unique common fixed point. We need a lemma to prove Theorem 2.2.

2.3 Lemma: Assume that the conditions in Theorem 2.2 hold.

Define a sequence $\{y_n\}$ in X by

$y_{2n} = Sx_{2n} = gx_{2n+1}$ and $y_{2n+1} = Tx_{2n+1} = fx_{2n+2}$

Then $d(y_{2n+1}, y_{2n}) \le \lambda\, d(y_{2n}, y_{2n-1})$.

Proof: Let $(\alpha, \beta) = d(y_{2n+1}, y_{2n})$ and $(\gamma, \delta) := d(y_{2n}, y_{2n-1})$. We show that $(\alpha, \beta) \le \lambda (\gamma, \delta)$

Now we have

$$\begin{aligned}
(\alpha, \beta) = d(y_{2n+1}, y_{2n}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\le \lambda \max \left\{ d(fx_{2n}, gx_{2n+1}), \frac{d(fx_{2n},Sx_{2n})+d(gx_{2n+1},,Tx_{2n+1})}{2}, \frac{d(fx_{2n},T_{2n+1})+d(gx_{2n+1},,Sx_{2n})}{2} \right\} \\
&= \lambda \max \left\{ d(y_{2n-1}, y_{2n}), \frac{d(,y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})}{2}, \frac{d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})}{2} \right\} \\
&\le \lambda \max \left\{ d(y_{2n-1}, y_{2n}), \frac{d(,y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})}{2}, \frac{d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})}{2} \right\} \\
&= \lambda \max \left\{ d(y_{2n-1}, y_{2n}), \frac{d(,y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})}{2} \right\} \\
&= \lambda \max \left\{ (\gamma, \delta), \left(\frac{\alpha+\gamma}{2}, \frac{\beta+\delta}{2}\right) \right\}
\end{aligned}$$

Therefore $(\alpha, \beta) \le \lambda \max\{ (\gamma, \delta), (\frac{\alpha+\gamma}{2}, \frac{\beta+\delta}{2})\} = \lambda (\max\{ \gamma, \frac{\alpha+\gamma}{2}\}, \max\{ \delta, \frac{\beta+\delta}{2}\}$. (2.3.2)

Write $\varepsilon = \max\left\{ \gamma, \frac{\alpha+\gamma}{2}\right\}$ *and* $\mu = \max \{\delta, \frac{\beta+\delta}{2}\}$.

We consider four cases to complete the proof.

Case 1: $\varepsilon = \gamma,\ \mu = \delta$

Case 2: $\varepsilon = \gamma,\ \mu = \frac{\beta+\delta}{2}$

Case 3: $\varepsilon = \frac{\alpha+\gamma}{2}, \mu = \delta$

Case 4: $\varepsilon = \frac{\alpha+\gamma}{2}, \mu = \frac{\beta+\delta}{2}$

Case 1: $\varepsilon = \gamma,\ \mu = \delta$

In this case we readily get $(\alpha, \beta) \leq \lambda\ (\gamma, \delta)$ as required.

Case 2: $\varepsilon = \gamma,\ \mu = \frac{\beta+\delta}{2}$

From (2.3.1) we have $(\alpha, \beta) \leq \lambda \max\ \{(\gamma, \frac{\beta+\delta}{2})$

This implies $\alpha \leq \lambda\gamma$ and $\beta \leq \lambda \frac{\beta+\delta}{2}$.

The second inequality implies $\beta(2 - \lambda) \leq \lambda\,\delta$. Since $\frac{1}{2-\lambda} < 1,\ so\ \beta \leq \lambda\,\delta$. Thus $(\alpha, \beta) \leq \lambda\ (\gamma, \delta)$

The remaining cases can be shown in the same manner.

Next we prove the main theorem.

Proof (Theorem 2.2):

From lemma 2.3 we have $d\ (y_{2n+1}, y_{2n}) \leq \lambda\ d\ (y_{2n}, y_{2n-1})$. By similar argument we can prove

$$d\ (y_{2n+1}, y_{2n+2}) \leq \lambda\ d\ (y_{2n}, y_{2n+1}).$$

Therefore $d(y_{n+1}, y_{n+2}) \leq \lambda\ d(y_n, y_{2n+1}) \leq \lambda^2\ d(y_{n-1}, y_{2n}) \leq \ldots \leq \lambda^{n+1}\ d(y_0, y_1)$

Now, for all m>n,

$$d\ (y_m, y_n) \leq\ d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_m, y_{m-1})$$

$$\leq\ \lambda^n\ d(y_0, y_1) +\ \lambda^{n+1}\ d(y_0, y_1) + \lambda^{m-1}\ d(y_0, y_1)$$

$$\leq \frac{\lambda^n}{1-\lambda}\ d\ (y_0, y_1)$$

This implies that $\lim_{n,m\to\infty} |d(y_m, y_n)|\ =0$. Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, so there exists z in X such that $y_n \to z$ in X (i.e., $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = ., \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = z$). Since T(X) is closed, so $z \in T(X)$.

Since $T(X) \subseteq f(X)$, so there is $u \in X$ such that z=fu. We now show Su=fu=z.

$$d(Su, z) \leq\ d(Su, Tx_{2n+1}) + d(Tx_{2n+1}, z)$$

$$\leq \lambda \max\ \{d(fu, gx_{2n+1}), \frac{d(fu, Su) + d(gx_{2n+1}, Tx_{2n+1})}{2},\ \ \frac{d(fu, T_{2n+1}) + d(gx_{2n+1}, Su)}{2}\}$$

Taking limit as n approaches to infinity, we get

$$d(Su, z) \leq \lambda \max\ \{d(z, z), \frac{d(z,Su) + d(z,z)}{2}, \frac{d(z,z) + d(z,Su)}{2}\}$$

$$= \lambda \max\ \{\ \frac{d(z,Su)}{2},\ \ \frac{d(z,Su)}{2}\}$$

$$= \lambda \frac{d(z,Su)}{2}$$

This implies d(Su, z)=0.Therefore, Su=z.

Since z=Su∈S(X) ⊆g(X), so there exist v in X such that z=gv. We can show that Tv = gv = z using an argument similar to the above. Therefore we get Su = fu = Tv = gv = z.

Claim: z is the unique common fixed point of f, g, S and T.

Since f and s are weakly compatible, so S fu= f Su. This implies Sz=fz.

Now, d(Sz, z) = d(Sz, Tv)

$$\leq \lambda \max \{d(fz, gv), \frac{d(fz,Sz)+d(gv,Tv)}{2}, \frac{d(fz,Tv)+d(gv,fz)}{2}\}$$

$$= \lambda \max \{d(Sz, gv), \frac{d(Sz,Sz)+d(z,z)}{2}, \frac{d(Sz,z)+d(z,Sz)}{2}\}= d(Sz, z)$$

This implies that d(Sz, z) = 0. Therefore, Sz = z = fz. By similar argument Tz = gz = z. Thus we proved that z is a common fixed point of f, g, S and T.

Let z, w be common fixed points of f, g, S and T.

Then d(z, w) = d(Sz, Tw)

$$\leq \lambda \max \{d(fz, gw), \frac{d(fz,Sz)+d(gw,Tw)}{2}, \frac{d(fz,Tw)+d(gw,Sz)}{2}\}$$

$$= \lambda \max \{d(z, w), \frac{d(z,z)+d(w,w)}{2}, \frac{d(z,w)+d(w,z)}{2}\}$$

$$= \lambda\, d(z, w)$$

This implies that d(z, w)=0 or w=z.

2.4 Remark: Theorem 2.1 is a corollary of Theorem 2.2 (our main result). The following example shows that this is a proper generalization. It is adapted from an example (see [7]) used to illustrate the corresponding result for metric spaces.

Example: Let X = {1, 2, 3, 4, 5} be a set. Define d: X×X → ℂ by:

d(1, 2) = d(1, 3)=d(3, 5) = (13/8, 13/8), d(1, 4)=d(1, 5)= d(2,4)=(7/4, 7/4), d(2,3)=d(4, 5)=(1,1), d(2, 5)=(15/8, 15/8) and d(3,4)=(2, 2).

Define F:X →X by F(1) = 1, F(2) = 4, F(3) = 4 , F(4) = 1 and f(5) = 2.

Now, with S=T=F and f=g=I(the identity mapping on X) we obtain

$d(Sx, Ty) \leq \frac{14}{15} \max \{d(fx, gy), \frac{d(fx,Sx)+d(gy,Ty)}{2}, \frac{d(fx,Ty)+d(gy,Sx)}{2}\}$ for all x, y in X.

But there does not exist nonnegative real numbers a, b and c satisfying a + 2b + 2c < 1 and

$d(Sx, Ty) \leq ad(fx, gy)+b[d(fx, Sx) + d(gy, Ty)] +c[d(fx, Ty) + d(gy, Sx)]$ simultaneously because, if such numbers exist,

$(\frac{3}{2}, \frac{3}{2})$ = d (1, 4) = d (F (1), F (2)) ≤ $a(\frac{13}{8}, \frac{13}{8}) + b(\frac{7}{4}, \frac{7}{4}) + c(\frac{25}{8}, \frac{25}{8})$

$(\frac{7}{4}, \frac{7}{4})$ = d (2, 4) = d (F (5), F (3)) ≤ $a(\frac{13}{8}, \frac{13}{8}) + b(\frac{31}{8}, \frac{31}{8}) + c(2, 2)$

This implies 26 ≤26a + 45b+ 41c ≤ 26(a+2b+2c) < 26, a contradiction.

Theorem 2.2 can be extended to vector valued metric spaces.

2.5 Theorem: Let (X, d) be a complete vector valued metric space and let f, g, S and T be four self-maps of X such that T(X) ⊆ f (X) and S(X) ⊆g(X) and satisfying

$$d(Sx, Ty) \leq \lambda \max \{ d(fx, gy), \frac{d(fx,Sx)+d(gy,Ty)}{2}, \frac{d(fx,Ty)+d(gy,Sx)}{2} \} \tag{2.5.1}$$

Suppose that the pairs {f, S} and {g, T} are weakly compatible and T(X) is closed. Then f, g, S and T have a unique common fixed point.

Proof: The same argument as Proof of Theorem2.2 and using a modification of lemma 2.3 provides the proof.

REFERENCES

[1] A. Azam, B.Fisher and M.Khan, *Common Fixed Point Theorems in Complex Valued Metric Spaces*, Numerical Functional Analysis and Optimization, 32(3) (2011), 243-253.

[2] S. Bhatt, S. Chaukiyal and R.C Dimiri, *A Common Fixed Point Theorem for Four Self maps in Complex Valued Metric Spaces*, J. Math. Anal. Appl., 1(3)(2011).

[3] S. Bhatt, S. Chaukiyal and R.C Dimiri, *Common Fixed Point of Mappings Satisfying Rational Inequality in Complex Valued Metric Spaces*, Intran. J. Pure & Appl. Math. Anal., 73 (2)(2011), 159-164.

[4] G.E Hardy and T.D Rogers, *A Generalization of A Fixed Point Theory of Reich*, Canad. Math. Bull., 16(2)(1973), 201-206.

[5] D.S Jaggi, *Some Unique Fixed Point Theorems,* Indian J. Pure. Appl. Math., 8(2) (1977), 223-230.

[6] G. Jungck and B.E Rhoades, *Fixed Points for Set Valued Functions without Continuity*, Indian. J. Pure Appl. Math., 29 (1998), 227-238.

[7] K.P.R Sastry and S.V.R Naidu, *Fixed Point Theorems for Generalized Contraction Mappings*, Yokohama Mathematical Journal, 28(1980)

[8] T. Zamifrescu, *Fixed Point Theorems in Metric Spaces*, Arch. Math. (Basel), 23(1972), 292-298.

Source of support: Nil, Conflict of interest: None Declared