International Journal of Mathematical Archive-3(7), 2012, 2651-2656
IJMA Available online through www.ijma.info ISSN 2229 – 5046

Uniquely Clean Idempotent 2×2 Matrices Over Integral Domains

K. N. Rajeswari
School of Mathematics, Vigyan Bhawan, Khandwa Road, INDORE--452017, India
&
Rafia Aziz*
Medi-Caps Institute of Technology and Management, A.B. Road, Pigdamber, INDORE–453331, India

(Received on: 07-07-12; Accepted on: 25-07-12)

ABSTRACT

Let R be a ring with identity. An element of R is said to be clean if it is the sum of a unit and an idempotent and it is uniquely clean if this representation is unique. It is well known that central idempotents in any ring are uniquely clean ([2]). In this paper it has been shown that if R is an Integral Domain then the central idempotents are the only uniquely clean idempotents in $M_2(R)$.

Keywords: *Uniquely clean, central idempotents, integral domain.*

2000 Mathematics Subject Classification: *16S50.*

1. INTRODUCTION

Let *R* be a ring with identity. Recall that an element *a* in *R* is called *clean* if it can be expressed as *a* = *e* + *u*, where *e* is an idempotent and *u* is a unit in *R*. It is said to be *uniquely clean* if this representation is unique. It is well known that idempotents in any ring are clean. If *e* is an idempotent in the ring, then *e* = (1 – *e*) + (2*e* – 1) is a clean representation of *e* since 1- *e* is an idempotent and 2*e* – 1 is a unit in that ring. Nicholson and Zhou [2, Example 1] have shown that central idempotents are uniquely clean in any ring.

Throughout this paper we let *R* be an Integral Domain and $M_2(R)$ be the ring of 2×2 matrices over *R*. It is well known that (see for example [1] or [3]) idempotents in $M_2(R)$ are:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \text{ (where } bc = a - a^2\text{)}$$

One can easily verify that out of these $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are the only central idempotents. In this paper we shall show that $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ (where $bc = a - a^2$) are not uniquely clean. This shows that non central idempotents are not uniquely clean. Hence uniquely clean idempotents are precisely the central idempotents in $M_2(R)$.

2. MAIN RESULTS

Theorem 2.1: Let *R* be an integral domain. In $M_2(R)$, idempotent matrix of the form $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ (where $bc = a - a^2$) is not uniquely clean.

Proof: For any idempotent matrix *E*, we always have *E* = (*I* - *E*) + (2*E* - *I*) as one clean representation (since, (*I* - *E*) is an idempotent and $(2E - I)^2$ = I). We shall call this the **natural clean representation** of *E*.

Corresponding author: *Rafia Aziz**
Medi-Caps Institute of Technology and Management, A.B. Road, Pigdamber, INDORE–453331, India

Consider the idempotent matrix $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ (where $bc = a - a^2$).

If, $a = 0$, then $bc = 0$ gives

$$E = \begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ w & 1 \end{bmatrix}$$

But

$$\begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & p \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & w-p \\ 0 & 1 \end{bmatrix}$$

where $\begin{bmatrix} 1 & p \\ 0 & 0 \end{bmatrix}$ is an idempotent and $\begin{bmatrix} -1 & w-p \\ 0 & 1 \end{bmatrix}$ is a unit with determinant -1 in $\mathrm{M}_2(\mathrm{R})$,

and $\begin{bmatrix} 0 & 0 \\ w & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ w-p & 1 \end{bmatrix}$

where $\begin{bmatrix} 1 & 0 \\ p & 0 \end{bmatrix}$ is an idempotent and $\begin{bmatrix} -1 & 0 \\ w-p & 1 \end{bmatrix}$ is a unit with determinant -1 in $\mathrm{M}_2(\mathrm{R})$. Hence both $\begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ w & 1 \end{bmatrix}$ are not uniquely clean.

Similarly if $a = 1$, then for $E = \begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix}$ we have the following clean representation,

$\begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & p \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & w-p \\ 0 & -1 \end{bmatrix}$ and for $E = \begin{bmatrix} 1 & 0 \\ w & 0 \end{bmatrix}$

we have the following clean representation,

$$\begin{bmatrix} 1 & 0 \\ w & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ p & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ w-p & -1 \end{bmatrix}$$

Hence both $\begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ w & 0 \end{bmatrix}$ are also not uniquely clean.

From now on $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ (where $bc = a - a^2 \neq 0$). In what follows we shall show that we can always express E as

$$E = E_1 + U$$

where $E_1 = \begin{bmatrix} x & y \\ w & 1-x \end{bmatrix}$ (with $yw = x - x^2$) is an idempotent and $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ is a unit with determinant -1 in $\mathrm{M}_2(\mathrm{R})$. This amounts to showing that there exist an idempotent matrix $E_1 = \begin{bmatrix} x & y \\ w & 1-x \end{bmatrix}$ (with $yw = x - x^2$) such that

$$det(E-E_1)=-1$$

or

$$det\left(\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}-\begin{bmatrix} x & y \\ w & 1-x \end{bmatrix}\right)=-1$$

or

$$det\left(\begin{bmatrix} a-x & b-y \\ c-w & -a+x \end{bmatrix}\right)=-1$$

This problem reduces to solving the equation

$$(a-x)(x-a)-(b-y)(c-w)=-1 \qquad \text{........(*)}$$

for *x*, *y* and *w* such that this solution $yw=x-x^2$.

Now from (*) we have

$$(2ax - a^2-x^2)-(bc-bw-yc+wy)=-1$$

using $bc=a-a^2$ and $yw=x-x^2$ we get

$$(2a-1)x-a+bw+yc=-1$$

Multiplying both sides by *y* and rearranging the terms we get

$$(2a-1)xy+y^2(c)+bwy+(-a+1)y=0$$

again using $yw=x-x^2$, we get

$$(-b)x^2+(2a-1)xy+(c)y^2+(b)x+(-a+1)y=0$$

Multiplying both sides by (-4*b*), we get

$$4b^2x^2+4b(1-2a)xy-4bcy^2-4b^2x+4b(a-1)y=0$$

or

$$(2bx+(1-2a)y-b)^2-((1-2a)y-b)^2-4bcy^2+4b(a-1)y=0$$

or

$$(2bx+(1-2a)y-b)^2-y^2-2by-b^2=0$$

or

$$(2bx+(1-2a)y-b)^2-(y+b)^2=0$$

Letting $x'=2bx+(1-2a)y-b$ and $y'=y+b$, we get

$$(x')^2-(y')^2=0$$

or

$$-(x')^2+(y')^2=0$$

or

$$(y'-x')(y'+x')=0 \qquad \text{....(**)}$$

© 2012, IJMA. All Rights Reserved

One of the parenthesis must be zero, say

$$(y' + x') = 0$$

or

$$y + b + 2bx + (1 - 2a)y - b = 0$$

or

$$2bx + (2 - 2a)y = 0$$

or

$$bx + (1 - a)y = 0$$

Clearly its general solution is

$x = (1 - a)t$, $y = -bt$

taking $t = 1 - a$, we get

$x = (1 - a)^2$, $y = -b(1 - a)$ such that

$$\begin{aligned} x(1-x) &= (1-a)^2(1-(1-a)^2) \\ &= (1-a)^2(-a)(a-2) \\ &= -bc(1-a)(a-2) \quad (\text{as } bc = a - a^2) \\ &= -b(1-a)[c(a-2)] \\ &= y(c(a-2)) \\ &= yw \quad (\text{letting } w = c(a-2)) \end{aligned}$$

Thus we get a solution $x = (1 - a)^2$, $y = -b(1 - a)$ and $w = c(a - 2)$ of eq. (*) satisfying $yw = x - x^2$ and hence an idempotent matrix

$$E_1 = \begin{bmatrix} (1-a)^2 & -b(1-a) \\ c(a-2) & 1-(1-a)^2 \end{bmatrix} = \begin{bmatrix} (1-a)^2 & -b(1-a) \\ c(a-2) & -a^2 + 2a \end{bmatrix}$$

which gives the following clean representation for *E*

$$\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} = \begin{bmatrix} (1-a)^2 & -b(1-a) \\ c(a-2) & -a^2 + 2a \end{bmatrix} + \begin{bmatrix} a-(1-a)^2 & 2b-2a \\ c-c(a-2) & (a-1)^2 - a \end{bmatrix}$$

where one can check that $det\left(\begin{bmatrix} a-(1-a)^2 & 2b-2a \\ c-c(a-2) & (a-1)^2 - a \end{bmatrix}\right) = -1$

From (**), we can also have

$$(y' - x') = 0$$

or

$$y + b - 2bx - (1 - 2a)y + b = 0$$

or

$$-2bx + 2ay + 2b = 0$$

or

© 2012, IJMA. All Rights Reserved

$bx+(-a)y=b$

Its general solution is

$x = 1 + at,\ \ y = bt$

taking $t = -a$, we get

$x=1-a^2\ ,\ y=-ba$

such that

$$\begin{aligned}
x(1-x) &= (1-a^2)(1-(1-a^2))\\
&= (1-a^2)(a^2)\\
&= (1-a)a[a(1+a)]\\
&= bc[a(1+a)]\\
&= (-ba)[-c(1+a)]\\
&= y[-c(1+a)]\\
&= yw \qquad\qquad (\text{letting } w = -c(1+a))
\end{aligned}$$

Hence we get another solution $x=1-a^2$, $y=-ba$ and $w=-c(1+a)$ for eq. (*) satisfying $yw=x-x^2$ and hence another idempotent matrix

$$E_1=\begin{bmatrix} 1-a^2 & -ba \\ -c(1+a) & 1-1+a^2 \end{bmatrix}=\begin{bmatrix} 1-a^2 & -ba \\ -c(1+a) & a^2 \end{bmatrix}$$

which gives the following clean representation for *E*

$$\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}=\begin{bmatrix} 1-a^2 & -ba \\ -c(1+a) & a^2 \end{bmatrix}+\begin{bmatrix} a-1+a^2 & b+ba \\ c+c(1+a) & 1-a-a^2 \end{bmatrix}$$

where one can check that $det\left(\begin{bmatrix} a-1+a^2 & b+ba \\ c+c(1+a) & 1-a-a^2 \end{bmatrix}\right)=-1$

Thus we get two different clean representations for *E*. Note that these clean representations are entirely different from the natural clean representation for *E* which is

$E = (I - E) + (2E - I)$

Or

$$\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}=\begin{bmatrix} 1-a & -b \\ -c & a \end{bmatrix}+\begin{bmatrix} 2a-1 & 2b \\ 2c & 1-2a \end{bmatrix}$$

Theorem 2.2: Let *R* be an integral domain. In $M_2(R)$, an idempotent is uniquely clean if and only if it is central. In other words, central idempotents are the only uniquely clean idempotents in $M_2(R)$.

Proof: The proof is clear from the above theorem and from the fact that central idempotents are uniquely clean [2].

REFERENCES

[1] J. Chen, X. Yang and Y. Zhou, On strongly clean matrix and triangular small matrix rings, Comm. Algebra 34(10)(2006) 3659-3674.

[2] W. K. Nicholson and Y. Zhou, Rings in which elements are uniquely the sum of an idempotent and a unit over commutative local rings, Glasgow Math. J. 46(2004) 227-236.

[3] K. N. Rajeswari and Rafia Aziz, A Note on Clean Matrices in $M_2(Z)$ International Journal of Algebra, Vol. 3(2009) No. 5, 241-248.

Source of support: Nil, Conflict of interest: None Declared