International Journal of Mathematical Archive-3(7), 2012, 2612-2621
IJMA Available online through www.ijma.info ISSN 2229 – 5046

THE LAPLACE HOMOTOPY ANALYSIS METHOD FOR HANDLING SYSTEM OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

V. G. Gupta[1] & Pramod Kumar[2*]

[1]Department of Mathematics, University of Rajasthan, Jaipur-302004, India
[2]Department of Mathematics, Jaipur National University, Jaipur-302024, India

(Received on: 03-07-12; Revised & Accepted on: 25-07-12)

ABSTRACT

In this paper, we present an algorithm of the Laplace homotopy analysis method (LHAM) to obtain the solutions for system of nonlinear fractional partial differential equations. The fractional derivatives are considered in Caputo sense. The proposed algorithm gives a procedure for constructing the set of base functions and gives the high order deformation equations in simple form. This method is applied to solve four systems of nonlinear fractional partial differential equations. Numerical results shows that LHAM is easy to implement and accurate when applied to solve system of equations.

Key words: *Systems of nonlinear fractional partial differential equations, Caputo derivatives, Laplace homotopy analysis method, homotopy analysis method.*

1. INTRODUCTION

Fractional order ordinary and partial differential equations as generalization of classical integer order ordinary differential equations are increasingly used to model problems in fluid flow, continuum and statistical mechanics, physics, engineering, economics, biology and other applications. Half order derivatives and integrals proved to be more useful for the formulation of certain electro-chemical problems then the classical model. A review of some applications of fractional calculus in continuum and statistical mechanics is given by Mainardi [1]. One of the most recent works on the subject of fractional calculus i.e. the theory of derivatives and integrals of fractional order, is the book of Podlubny [5], which deals principally with fractional differential equations and today there are many works on fractional calculus. For most of fractional differential equations there exist no method that yield on exact solution of fractional differential equations, so approximation or numerical techniques must be used, such as Adomian decomposition method [2,4,17,18,21], variation iteration method [17,20,26,27], homotopy perturbation method [7,11,12,19,27], homotopy perturbation transform method [24] and so on. However region of convergence of the corresponding result is rather small as shown in this paper.

Recently, Liao [15] proposed a powerful analytical method, namely the homotopy analysis method (HAM) for solving linear and nonlinear differential and integral equations. Different from perturbation techniques the HAM does not depend upon small or large parameters. A systematic and clear exposition on HAM is given in [16]. This method has been successfully applied to solve many types of nonlinear problems such as Riccati differential equation with fractional order [6], nonlinear Vakhnenko equation [25], the Glauret-jet problem [22], fractional KdV-Bergers - Kuromoto equation [8] and so on. The objective of present paper is to apply the Laplace homotopy analysis method [10] to provide symbolic approximate solutions for the system of linear and nonlinear partial differential equations of fractional order. The Laplace homotopy analysis method is the combination of HAM and Laplace transform. The organization of this paper is as follows: A brief review of the fractional calculus is given in next section. The Laplace homotopy analysis method is presented in section 3; four numerical examples are given to show the applicability of the considered method in section 4.

2. BASIC DEFINITIONS

For the concept of fractional derivatives, we will adopt Caputo's definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variable and their integral order which is the case in most physical processes. Some basic definitions and properties of fractional calculus theory which we have used in this paper are given in this section.

Corresponding author: *Pramod Kumar[2*]*
[2]Department of Mathematics, Jaipur National University, Jaipur-302024, India

Definition 2.1: A real function f(x), x > 0 is said to be in the space C_μ, $\mu \in R$, if there exist a real number p (>μ) such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C\,[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu, m \in N \cup \{0\}$.

Definition 2.2:.The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1} f(t)\,dt \;, \quad \alpha > 0,\; x > 0 \tag{1}$$

$$J^0 f(x) = f(x) \tag{2}$$

Properties of the operator J^α can be found in [9, 13], we mention only the following:

(i) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$

(ii) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$

(iii) $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$.

Definition 2.3:.The fractional derivative of f(x) in the Caputo sense is defined as [9]

$$D_*^\alpha f(x) = J^{m-\alpha} D_*^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt \tag{3}$$

For $m-1 < \alpha \leq m,\; m \in N,\; x > 0,\; f \in C_{-1}^m$.

Also, we need here three basic properties

(i) $D_*^\alpha J^\alpha f(x) = f(x)$

(ii) $J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\dfrac{x^k}{k!}, \qquad x > 0$

(iii) $D_*^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}\;;\; x > 0, \gamma > 0.$

For $m-1 < \alpha \leq m, m \in N, \mu \geq -1$ and $f \in C_\mu^m$.

Lemma 2.1:.If $m-1 < \alpha \leq m, m \in N,$ then the Laplace transform of the fractional derivative $D_*^\alpha f(t)$ is

$$L\,(D_*^\alpha f(t)) = s^\alpha \bar{f}(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\, s^{\alpha-k-1},\; t > 0 \tag{4}$$

Where $\bar{f}(s)$ is the Laplace transform of f (t).

3. LAPLACE HOMOTOPY ANALYSIS METHOD

The homotopy analysis method which provides an analytic approximate solution is applied to various non linear problems. They use the auxiliary linear operator to be D_t^α .In this section we extend the applications of Laplace homotopy analysis method [10] to the following system of fractional differential equations:

$$D_t^\alpha u_i(x,t) = f(t, u_i(x,t), u_{ix}(x,t)) \qquad ,t>0,\ 0<\alpha\le 1 \tag{5}$$

Subject to the initial conditions

$$u_i(x,0) = a_i \qquad ;i = 1,2,3,\dots,n \tag{6}$$

Applying the Laplace transform of both sides of equation (5), we have

$$L\left(D_t^\alpha u_i(x,t)\right) = L\left(f(t,u_i(x,t),u_{ix}(x,t)\right)$$

Using (6), then we have

$$s^\alpha \overline{u_i}(x,s) - s^{\alpha-1}a_i = L(f(t,u_i(x,t),u_{ix}(x,t))$$

$$\overline{u_i}(x,s) = \frac{a_i}{s} + \frac{1}{s^\alpha} L(f(t,u_i(x,t),u_{ix}(x,t)) \tag{7}$$

Where $L(u_i(x,t)) = \overline{u_i}(x,s)$

The so-called zeroth-order deformation equation of the Laplace equation (7) has the form

$$(1-q)\left[\overline{\phi}_i(x,s,q) - \overline{u_{i0}}(x,s)\right] = q\, h_i \left[\overline{\phi}_i(x,s,q) - \frac{a_i}{s} - \frac{1}{s^\alpha} L(f(t,\phi_i(x,t,q), \frac{d}{dx}\phi_i(x,t,q)))\right] \tag{8}$$

Where q ϵ [0, 1] is an embedding parameter, when q=0 and q=1, we have $\overline{\phi}_i(x,s,0) = \overline{u_{i0}}(x,s)$ and $\overline{\phi}_i(x,s,1) = \overline{u_i}(x,s)$. Thus as q increases from 0 to 1, $\overline{\phi}_i(x,s,q)$ varies from $\overline{u_{i0}}(x,s)$ *to* $\overline{u_i}(x,s)$. Expanding $\overline{\phi}_i(x,s,q)$ in Taylor series with respect to q, one has

$$\overline{\phi}_i(x,s,q) = \overline{u_{i0}}(x,s) + \sum_{m=1}^{\infty} \overline{u_{im}}(x,s)\, q^m \tag{9}$$

Where $\overline{u_{im}}(x,s) = \dfrac{1}{m!}\dfrac{\partial^m}{\partial q^m}\overline{\phi}_i(x,s,q)\ \Big|_{q=0}$ (10)

If the auxiliary parameter h_i and the initial guesses $\overline{u_{i0}}(x,s)$ are so properly chosen, then the series (9) is converses at q=1 and one has

$$\overline{u_i}(x,s) = \overline{u_{i0}}(x,s) + \sum_{m=1}^{\infty} \overline{u_{im}}(x,s) \tag{11}$$

Define the vectors

$$\overrightarrow{\overline{u_{im}}}(x,s) = \left\{\overline{u_{i0}}(x,s), \overline{u_{i1}}(x,s), \overline{u_{i2}}(x,s), \dots \overline{u_{im}}(x,s)\right\}$$

Differentiating equation (8) m times with respect to embedding parameter q, then setting q=0, $h_i = -1$ and finally dividing that by m! , we have the so-called mth –order deformation equation

$$\overline{u_{im}}(x,s) = \chi_m \overline{u_{i(m-1)}}(x,s) - \Re_{im}(\overrightarrow{\overline{u_{i(m-1)}}}(x,s)) \tag{12}$$

where

$$\Re_{im}(\vec{\bar{u}}_{i(m-1)}(x,s)) = \bar{u}_{i(m-1)}(x,s) - \frac{1}{s^{\alpha}}\left(\frac{1}{m-1!}\frac{\partial^{m-1}}{\partial q^{m-1}}[f(t,\phi(x,t,q),\frac{d}{dx}\phi(x,t,q)\right)_{q=0} - \frac{a_i}{s}(1-\chi_m) \quad (13)$$

And $$\chi_m = \begin{cases} 0 & , m \le 1 \\ 1 & , m > 1 \end{cases} \quad (14)$$

Applying the inverse Laplace transforms of (12), then we have the power series solution

$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of (5)

4. NUMERICAL EXAMPLES

In order to assess the advantage and the accuracy of the Laplace homotopy analysis method presented in this paper for linear and nonlinear partial differential equations of fractional order, we applied it to the following problems:

Example 4.1: Consider the following time fractional non homogeneous nonlinear system [23]

$$\begin{aligned} D_t^{\alpha} u - u_x v - u &= 1 \\ D_t^{\beta} v + u v_x + v &= 1 \end{aligned} \; ; 0 < \alpha, \beta \le 1 \quad (15)$$

Subject to the initial conditions

$$u(x,0) = e^{-x} \quad , v(x,0) = e^{x} \quad (16)$$

Taking the Laplace transform of both sides of (15) and by using (16), we have

$$\begin{aligned} \bar{u}(x,s) - \frac{1}{s^{\alpha}} L(u_x v) - \frac{1}{s^{\alpha}} \bar{u}(x,s) - \frac{1}{s} e^{-x} - \frac{1}{s^{\alpha+1}} &= 0 \\ \bar{v}(x,s) + \frac{1}{s^{\beta}} L(u v_x) + \frac{1}{s^{\beta}} \bar{v}(x,s) - \frac{1}{s} e^{x} - \frac{1}{s^{\beta+1}} &= 0 \end{aligned} \quad (17)$$

Then in view of (12) and (13), we have

$$\overline{u_m}(x,s) = \chi_m \overline{u_{m-1}}(x,s) - \left(\overline{u_{m-1}}(x,s) - \frac{1}{s^{\alpha}} L\left(\sum_{j=0}^{m-1} u_{jx} v_{m-1-j}\right) - \frac{1}{s^{\alpha}} \overline{u_{m-1}}(x,s) - (\frac{1}{s} e^{-x} + \frac{1}{s^{\alpha+1}})(1-\chi_m)\right)$$

$$\overline{v_m}(x,s) = \chi_m \overline{v_{m-1}}(x,s) - \left(\overline{v_{m-1}}(x,s) + \frac{1}{s^{\beta}} L\left(\sum_{j=0}^{m-1} u_j v_{(m-1-j)x}\right) + \frac{1}{s^{\beta}} \overline{v_{m-1}}(x,s) - (\frac{1}{s} e^{x} + \frac{1}{s^{\beta+1}})(1-\chi_m)\right) \quad (18)$$

$$\overline{u_0}(x,s) = \frac{1}{s} e^{-x} + \frac{1}{s^{\alpha+1}}$$

$$\overline{v_0}(x,s) = \frac{1}{s} e^{x} + \frac{1}{s^{\beta+1}}$$

$$\overline{u_1}(x,s) = \frac{1}{s^{\alpha+1}} e^{-x} + \frac{1}{s^{2\alpha+1}} - \frac{1}{s^{\alpha+1}} - \frac{1}{s^{\alpha+\beta+1}} e^{-x}$$

$$\overline{v_1}(x,s) = -\frac{1}{s^{\beta+1}} e^{x} - \frac{1}{s^{2\beta+1}} - \frac{1}{s^{\beta+1}} - \frac{1}{s^{\alpha+\beta+1}} e^{x}$$

$$\overline{u_2}(x,s) = \frac{1}{s^{\alpha+\beta+1}}e^{-x} + \frac{2}{s^{2\alpha+\beta+1}} + \frac{1}{s^{\alpha+\beta+1}} + \frac{1}{s^{\alpha+2\beta+1}}e^{-x} - \frac{2}{s^{2\alpha+1}} - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{1}{s^{2\alpha+\beta+1}}e^{-x}$$
$$+\frac{\Gamma(\alpha+2\beta+1)}{\Gamma(\alpha+\beta+1)\Gamma(\beta+1)}\frac{1}{s^{2\alpha+2\beta+1}}e^{-x} + \frac{1}{s^{2\alpha+1}}e^{-x} + \frac{1}{s^{3\alpha+1}} - \frac{1}{s^{2\alpha+\beta+1}}e^{-x}$$

$$\overline{v_2}(x,s) = \frac{1}{s^{\alpha+\beta+1}}e^{x} + \frac{2}{s^{\alpha+2\beta+1}} - \frac{1}{s^{\alpha+\beta+1}} - \frac{1}{s^{2\alpha+\beta+1}}e^{x} + \frac{2}{s^{2\beta+1}} + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{1}{s^{\alpha+2\beta+1}}e^{x}$$
$$+\frac{\Gamma(2\alpha+\beta+1)}{\Gamma(\alpha+\beta+1)\Gamma(\beta+1)}\frac{1}{s^{2\alpha+2\beta+1}}e^{x} + \frac{1}{s^{2\beta+1}}e^{x} + \frac{1}{s^{3\beta+1}} + \frac{1}{s^{\alpha+2\beta+1}}e^{x}$$

Now

$$\bar{u}(x,s) = \overline{u_0}(x,s) + \overline{u_1}(x,s) + \overline{u_2}(x,s) + \ldots$$
$$\bar{v}(x,s) = \overline{v_0}(x,s) + \overline{v_1}(x,s) + \overline{v_2}(x,s) + \ldots \tag{19}$$

Taking the inverse Laplace transform of both sides of (19), we have

$$u(x,t) = e^{-x} + e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - e^{-x}\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots.$$
$$v(x,t) = e^{x} - e^{x}\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} - e^{x}\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + e^{x}\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \ldots. \tag{20}$$

If we take α = β = 1, we have

$$u(x,t) = e^{-x}(1 + t + \frac{t^2}{2!} + \ldots) \quad = e^{-x+t}$$
$$v(x,t) = e^{x}(1 - t + \frac{t^2}{2!} + \ldots) \quad = e^{x-t} \tag{21}$$

The result we have found now is the same with [23]

Example 4.2: Consider the following system of linear fractional PDEs

$$\begin{aligned} D_t^{\alpha}u - v_x + u + v &= 0 \\ D_t^{\beta}v - u_x + v + u &= 0 \end{aligned} \qquad 0 < \alpha, \beta \le 1 \tag{22}$$

Subject to the initial conditions

$$u(x,0) = \sinh x \quad , \quad v(x,0) = \cosh x \tag{23}$$

Taking the Laplace of both sides of the equation (22) and using (23), we have

$$\bar{u}(x,s) - \frac{1}{s^{\alpha}}\left(L\left(v_x\right) - \bar{u}(x,s) - \bar{v}(x,s) \right) - \frac{1}{s}\sinh x = 0$$
$$\bar{v}(x,s) - \frac{1}{s^{\beta}}\left(L\left(u_x\right) - \bar{u}(x,s) - \bar{v}(x,s) \right) - \frac{1}{s}\cosh x = 0 \tag{24}$$

Now in view of equation (12) and (13), we have

$$\overline{u_m}(x,s) = \chi_m\overline{u_{m-1}}(x,s) - \left(\overline{u_{m-1}}(x,s) - \frac{1}{s^{\alpha}}\left(L\left(v_{(m-1)x}\right) - \overline{u_{m-1}}(x,s) - \overline{v_{m-1}}(x,s) \right) - \frac{1}{s}\sinh x(1-\chi_m)\right)$$
$$\overline{v_m}(x,s) = \chi_m\overline{v_{m-1}}(x,s) - \left(\overline{v_{m-1}}(x,s) - \frac{1}{s^{\alpha}}\left(L\left(u_{(m-1)x}\right) - \overline{u_{m-1}}(x,s) - \overline{v_{m-1}}(x,s) \right) - \frac{1}{s}\cosh x(1-\chi_m)\right) \tag{25}$$

$$\overline{u_0}(x,s)=\frac{1}{s}\sinh x$$

$$\overline{v_0}(x,s)=\frac{1}{s}\cosh x$$

$$\overline{u_1}(x,s)=-\frac{1}{s^{\alpha+1}}\cosh x$$

$$\overline{v_1}(x,s)=-\frac{1}{s^{\beta+1}}\sinh x$$

$$\overline{u_2}(x,s)=-\frac{1}{s^{\alpha+\beta+1}}\cosh x+\frac{1}{s^{2\alpha+1}}\cosh x+\frac{1}{s^{\alpha+\beta+1}}\sinh x$$

$$\overline{v_2}(x,s)=-\frac{1}{s^{\alpha+\beta+1}}\sinh x+\frac{1}{s^{2\beta+1}}\sinh x+\frac{1}{s^{\alpha+\beta+1}}\cosh x$$

$$\overline{u_3}(x,s)=-\frac{1}{s^{2\alpha+\beta+1}}\cosh x-\frac{1}{s^{3\alpha+1}}\cosh x+\frac{1}{s^{2\alpha+\beta+1}}\sinh x$$

$$\overline{v_3}(x,s)=-\frac{1}{s^{\alpha+2\beta+1}}\cosh x-\frac{1}{s^{3\beta+1}}\sinh x+\frac{1}{s^{\alpha+2\beta+1}}\sinh x$$

So

$$\bar{u}(x,s)=\overline{u_0}(x,s)+\overline{u_1}(x,s)+\overline{u_2}(x,s)+\ldots$$
$$\bar{v}(x,s)=\overline{v_0}(x,s)+\overline{v_1}(x,s)+\overline{v_2}(x,s)+\ldots \tag{26}$$

Now taking the inverse Laplace transform of both sides of equation (26), we have

$$u(x,t)=\sinh x\left\{1+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\ldots\right\}-\cosh x\left\{\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right\}$$
$$v(x,t)=\cosh x\left\{1+\frac{t^{2\beta}}{\Gamma(2\beta+1)}+\frac{t^{4\beta}}{\Gamma(4\beta+1)}+\ldots\right\}-\sinh x\left\{\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{3\beta}}{\Gamma(3\beta+1)}+\ldots\right\} \tag{27}$$

If we take α = β = 1, we have

$$u(x,t)=\sinh x\left\{1+\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots\right\}-\cosh x\left\{t+\frac{t^3}{3!}+\ldots\right\}$$
$$v(x,t)=\cosh x\left\{1+\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots\right\}-\sinh x\left\{t+\frac{t^3}{3!}+\ldots\right\} \tag{28}$$

The result we have found now is the same with [3]

Example 4.3: Consider the following system of fractional PDEs

$$D_t^{\alpha}u-D_x^{\beta}v+u+v=0 \qquad 0<\alpha,\beta\le 1$$
$$D_t^{\alpha}v-D_x^{\beta}u+v+u=0 \tag{29}$$

Subject to the initial conditions

$$u(x,0)=\sinh x\ ,\ v(x,0)=\cosh x \tag{30}$$

Taking the Laplace transform of both sides of (29) and using (30), we have

$$\bar{u}(x,s)-\frac{1}{s^{\alpha}}\left(\ L(D_x^{\beta}v)-\bar{u}(x,s)-\bar{v}(x,s)\ \right)-\frac{1}{s}\sinh x=0$$
$$\bar{v}(x,s)-\frac{1}{s^{\alpha}}\left(\ L(D_x^{\beta}u)-\bar{u}(x,s)-\bar{v}(x,s)\ \right)-\frac{1}{s}\cosh x=0 \tag{31}$$

In view of the equation (12) and (13), we have

$$\overline{u_m}(x,s) = \chi_m \overline{u_{m-1}}(x,s) - \left(\overline{u_{m-1}}(x,s) - \frac{1}{s^{\alpha}}\left(L(D_x^{\beta} v_{m-1}) - \overline{u_{m-1}}(x,s) - \overline{v_{m-1}}(x,s) \right) - \frac{1}{s}\sinh x(1-\chi_m) \right)$$
$$\overline{v_m}(x,s) = \chi_m \overline{v_{m-1}}(x,s) - \left(\overline{v_{m-1}}(x,s) - \frac{1}{s^{\alpha}}\left(L(D_x^{\beta} u_{m-1}) - \overline{u_{m-1}}(x,s) - \overline{v_{m-1}}(x,s) \right) - \frac{1}{s}\cosh x(1-\chi_m) \right) \quad (32)$$

So

$$\overline{u_0}(x,s) = \frac{1}{s}\sinh x$$
$$\overline{v_0}(x,s) = \frac{1}{s}\cosh x$$

$$\overline{u_1}(x,s) = \frac{1}{s^{\alpha+1}}(f_1(x) - \sinh x - \cosh x)$$
$$\overline{v_1}(x,s) = \frac{1}{s^{\alpha+1}}(g_1(x) - \sinh x - \cosh x)$$
$$\overline{u_2}(x,s) = \frac{1}{s^{2\alpha+1}}(g_2(x) - 2g_1(x) - 2f_1(x) + 2\sinh x + 2\cosh x)$$
$$\overline{v_2}(x,s) = \frac{1}{s^{2\alpha+1}}(f_2(x) - 2g_1(x) - 2f_1(x) + 2\sinh x + 2\cosh x)$$
$$\overline{u}(x,s) = \overline{u_0}(x,s) + \overline{u_1}(x,s) + \overline{u_2}(x,s) + \ldots$$
$$v(x,s) = \overline{v_0}(x,s) + \overline{v_1}(x,s) + \overline{v_2}(x,s) + \ldots \quad (33)$$

Now taking the inverse Laplace transform of both side of equation (33), we have

$$u(x,t) = \sinh x + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(f_1(x) - \sinh x - \cosh x \right) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left(g_2(x) - 2f_1(x) - 2g_1(x) + 2\sinh x + 2\cosh x \right) + \ldots$$
$$v(x,t) = \cosh x + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(g_1(x) - \sinh x - \cosh x \right) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left(f_2(x) - 2f_1(x) - 2g_1(x) + 2\sinh x + 2\cosh x \right) + \ldots \quad (34)$$

where

$$f_1(x) = \frac{x^{-\beta}}{\Gamma(1-\beta)} + \frac{x^{2-\beta}}{\Gamma(3-\beta)} + \frac{x^{4-\beta}}{\Gamma(5-\beta)} + \ldots$$
$$f_2(x) = \frac{x^{-2\beta}}{\Gamma(1-2\beta)} + \frac{x^{2-2\beta}}{\Gamma(3-2\beta)} + \frac{x^{4-2\beta}}{\Gamma(5-2\beta)} + \ldots$$
$$g_1(x) = \frac{x^{1-\beta}}{\Gamma(2-\beta)} + x^{3-\beta}\, \frac{x^{3-\beta}}{\Gamma(4-\beta)} + \frac{x^{5-\beta}}{\Gamma(6-\beta)} + \ldots$$
$$g_2(x) = \frac{x^{1-2\beta}}{\Gamma(2-2\beta)} + \frac{x^{3-2\beta}}{\Gamma(4-2\beta)} + \frac{x^{5-2\beta}}{\Gamma(6-2\beta)} + \ldots$$

The result we have found now is the same with [14] for $h_1 = h_2 = -1$

Example 4.4:.Consider the following system of fractional PDEs

$$D_t^{\alpha} u + v_x w_y - v_y w_x = -u$$
$$D_t^{\beta} v + w_x u_y + w_y u_x = v$$
$$D_t^{\gamma} w + u_x v_y + u_y v_x = w \tag{35}$$

Subject to the initial conditions

$$u(x,y,0) = e^{x+y} \ , \ v(x,y,0) = e^{x-y} \ , w(x,y,0) = e^{-x+y} \tag{36}$$

Taking the Laplace transform of both sides of equations (35) and using (36), we have

$$\bar{u}(x,y,s) + \frac{1}{s^{\alpha}}\left(\ L(v_x w_y - v_y w_x + u) \ \right) - \frac{1}{s} e^{x+y} = 0$$
$$\bar{v}(x,y,s) + \frac{1}{s^{\beta}}\left(\ L(w_x u_y + w_y u_x - v) \ \right) - \frac{1}{s} e^{x-y} = 0$$
$$\bar{w}(x,y,s) + \frac{1}{s^{\gamma}}\left(\ L(u_x v_y + u_y v_x - w) \ \right) - \frac{1}{s} e^{-x+y} = 0 \tag{37}$$

Now in view of the equation (12) and (13), we have

$$\overline{u_m}(x,y,s) = \chi_m \overline{u_{m-1}}(x,y,s) - \left(\overline{u_{m-1}}(x,y,s) + \frac{1}{s^{\alpha}}\left(\ L\left(\sum_{j=0}^{m-1} v_{jx} w_{(m-1-j)y} - \sum_{j=0}^{m-1} v_{jy} w_{(m-1-j)x} + u_{m-1} \right) \ \right) - \frac{1}{s} e^{x+y}(1-\chi_m) \right)$$
$$\overline{v_m}(x,y,s) = \chi_m \overline{v_{m-1}}(x,y,s) - \left(\overline{v_{m-1}}(x,y,s) + \frac{1}{s^{\beta}}\left(\ L\left(\sum_{j=0}^{m-1} w_{jx} u_{(m-1-j)y} - \sum_{j=0}^{m-1} w_{jy} u_{(m-1-j)x} - v_{m-1} \right) \ \right) - \frac{1}{s} e^{x-y}(1-\chi_m) \right)$$
$$\overline{w_m}(x,y,s) = \chi_m \overline{w_{m-1}}(x,y,s) - \left(\overline{w_{m-1}}(x,y,s) + \frac{1}{s^{\gamma}}\left(\ L\left(\sum_{j=0}^{m-1} u_{jx} v_{(m-1-j)y} - \sum_{j=0}^{m-1} u_{jy} v_{(m-1-j)x} - w_{m-1} \right) \ \right) - \frac{1}{s} e^{-x+y}(1-\chi_m) \right) \tag{38}$$

$$\overline{u_0}(x,y,s) = \frac{1}{s} e^{x+y}$$
$$\overline{v_0}(x,y,s) = \frac{1}{s} e^{x-y}$$
$$\overline{w_0}(x,y,s) = \frac{1}{s} e^{-x+y}$$
$$\overline{u_1}(x,y,s) = -\frac{1}{s^{\alpha+1}} e^{x+y}$$
$$\overline{v_1}(x,y,s) = \frac{1}{s^{\beta+1}} e^{x-y}$$
$$\overline{w_1}(x,y,s) = \frac{1}{s^{\gamma+1}} e^{-x+y}$$
$$\overline{u_2}(x,y,s) = \frac{1}{s^{2\alpha+1}} e^{x+y}$$
$$\overline{v_2}(x,y,s) = \frac{1}{s^{2\beta+1}} e^{x-y}$$
$$\overline{w_2}(x,y,s) = \frac{1}{s^{2\gamma+1}} e^{-x+y}$$

$$\bar{u}(x,y,s) = \overline{u_0}(x,y,s) + \overline{u_1}(x,y,s) + \overline{u_2}(x,y,s) + \ldots$$
$$\bar{v}(x,y,s) = \overline{v_0}(x,y,s) + \overline{v_1}(x,y,s) + \overline{v_2}(x,y,s) + \ldots$$
$$\bar{w}(x,y,s) = \overline{w_0}(x,y,s) + \overline{w_1}(x,y,s) + \overline{w_2}(x,y,s) + \ldots \tag{39}$$

Now taking the inverse Laplace transform of both sides of (39), we have

$$u(x,y,t) = e^{x+y}\left\{1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right\}$$
$$v(x,y,t) = e^{x-y}\left\{1 + \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \ldots\right\}$$
$$w(x,y,t) = e^{-x+y}\left\{1 + \frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \ldots\right\} \tag{40}$$

If we take $\alpha = \beta = \gamma = 1$, then we have

$$u(x,y,t) = e^{x+y-t}\ ,\ v(x,y,t) = e^{x-y-t}\ ,\ w(x,y,t) = e^{-x+y-t} \tag{41}$$

The result we have now is the same with [3]

CONCLUSION

In this paper, Laplace homotopy analysis method which is based on homotopy analysis method and Laplace transform is used to solve the system of partial differential equations of fractional order. The nonlinear term can be easily handled by the use of m^{th} -order deformation equations of HAM. Different from the other analytical methods, it provides us a simple way to adjust and control the convergence region of solution in series by choosing proper value for auxiliary parameters h and auxiliary linear operator. Also we can see that the homotopy perturbation transform method is the special case of Laplace homotopy analysis method. Finally, generally speaking, the proposed approach can be further implemented to solve other nonlinear problems in fractional calculus field.

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Source of support: Nil, Conflict of interest: None Declared