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SOME NEW OPERATORS ON INTERVAL FUZZY NUMBERS AND INTERVAL FUZZY NUMBER MATRICES

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ABSTRACT

In this paper, some new elementary operators on Interval Fuzzy Numbers (IFNs) and some new operators on Interval Fuzzy Number Matrices (IFNMs) are defined. Using these operators, some important properties are proved.

INTRODUCTION

Real world decision making problems are very often uncertain (or) vague in a number of ways. In 1965, Zadeh [6] introduced the concept of fuzzy set theory to meet those problems. The fuzzyness can be represented by different ways. One of the most useful representation is the membership function. Depending on the nature of the membership function the fuzzy numbers can be classified in different forms, such as Triangular Fuzzy Numbers (TFNs), Trapezoidal Fuzzy Numbers, Interval Fuzzy Numbers etc. Fuzzy matrices play an important role in scientific development. Fuzzy matrices were introduced by M.G. Thomason [5]. A.K. Shymal and M. Pal introduced Triangular Fuzzy Number Matrices [2]. Interval Fuzzy Number Matrices are introduced by M. Pal, Gobinda Murmu and Anita Pal [4]. Two new operators and some properties of fuzzy matrices over these new operators are given in [1]. Some new operators on Triangular Fuzzy Numbers (TFNs) and Triangular Fuzzy Number Matrices (TFNMs) are discussed in [3].

In this paper, some new elementary operators on Interval Fuzzy Numbers (IFNs) and some new operators on Interval Fuzzy Number Matrices (IFNMs) are defined. Using these operators, some important properties are proved.

1. BASIC DEFINITIONS

Definition: 1.1 An **Interval number** is defined as $\hat{A} = [a_L, a_R]$ ={a: $a_L \leq a \leq a_R$ } where a_L and a_R are the real numbers called the left end point and right end point respectively of the interval $\hat{A}$.

Example: 1.2 $\hat{A}$ = [3, 6] is an interval number.

Definition: 1.3 A matrix of order nxn is said to be an **interval matrix** if all its elements are the interval numbers.

Definition: 1.4 An **interval fuzzy number** is defined as $\hat{A}$ =[a_L, a_R]={a: $a_L \leq a \leq a_R$ } where a_L , $a_R \in$[0,1] and are called the left end point and right end point respectively of the interval $\hat{A}$.

Example: 1.5 $\hat{M}$ = [0.65, 0.90] is an interval fuzzy number.

Definition: 1.6 A matrix of order nxn is said to be an **interval fuzzy number matrix (IFNM)** if all its elements are the interval fuzzy numbers.

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Example: 1.7 M = $\begin{bmatrix} [0.5,0.1] & [0.1,0.7] \\ [0.3,0.2] & [0.6,0.4] \end{bmatrix}$ is an interval fuzzy number matrix.

2. SOME NEW OPERATORS ON INTERVAL FUZZY NUMBERS AND INTERVAL FUZZY NUMBER MATRICES

Definition: 2.1 Let $\hat{M} = [m_L, m_R]$ and $\hat{N} = [n_L, n_R]$ be two interval fuzzy numbers where $m_L, m_R, n_L, n_R \in [0,1]$. Then the following operators are defined.

(1) $\hat{M} \oplus \hat{N} = [m_L + n_L - m_L n_L, m_R + n_R - m_R n_R]$,

(2) $\hat{M} \odot \hat{N} = [m_L n_L, m_R n_R]$

(3) $\hat{M} \vee \hat{N} = [m_L \vee n_L, m_R \vee n_R]$ where x ∨ y means max {x, y}. That is, x ∨ y = max {x, y}.

(4) $\hat{M} \wedge \hat{N} = [m_L \wedge n_L, m_R \wedge n_R]$ where x ∧ y means min {x, y}. That is, x ∧ y = min {x, y}.

(5) $\hat{M} \ominus \hat{N} = [m_L \ominus n_L, m_R \ominus n_R]$ where $x \ominus y = \begin{cases} x, & \text{if } x > y \\ 0, & \text{if } x \le y \end{cases}$

(6) $\hat{M} \ge \hat{N}$ if $m_L \ge n_L$ and $m_R \ge n_R$.

(7) $\hat{M} \pm \hat{N} = [m_L \pm n_L, m_R \pm n_R]$.

Definition: 2.2 Let M = $(\hat{M}_{ij})_{mxn}$ and N = $(\hat{N}_{ij})_{mxn}$ be two interval fuzzy number matrices of the same order where $\hat{M}_{ij} = [m_{L_{ij}}, m_{R_{ij}}]$ and $\hat{N}_{ij} = [n_{L_{ij}}, n_{R_{ij}}]$. Then the following operators are defined.

(1) M ⊕ N = $(\hat{M}_{ij} \oplus \hat{N}_{ij})$

(2) M ⊙ N = $(\hat{M}_{ij} \odot \hat{N}_{ij})$

(3) M ∨ N = $(\hat{M}_{ij} \vee \hat{N}_{ij})$

(4) M ∧ N = $(\hat{M}_{ij} \wedge \hat{N}_{ij})$

(5) M ⊖ N = $(\hat{M}_{ij} \ominus \hat{N}_{ij})$

(6) M ≥ N iff $\hat{M}_{ij} \ge \hat{N}_{ij}$ ∀ i = 1 to m, j = 1 to n.

Property: 2.3 For any interval fuzzy number matrix M,

(i) M ⊕ M ≥ M
(ii) M ⊙ M ≤ M

Proof:
(i) The ij[th] element of M ⊕ M is $\hat{M}_{ij} \oplus \hat{M}_{ij}$ which is equal to

$[m_{L_{ij}} + m_{L_{ij}} - m_{L_{ij}} . m_{L_{ij}}, m_{R_{ij}} + m_{R_{ij}} - m_{R_{ij}} . m_{R_{ij}}]$

Consider

$$m_{L_{ij}} + m_{L_{ij}} - m_{L_{ij}} . m_{L_{ij}} = m_{L_{ij}} + m_{L_{ij}} (1 - m_{L_{ij}}) \ge m_{L_{ij}} \quad (1)$$

$$m_{R_{ij}} + m_{R_{ij}} - m_{R_{ij}} . m_{R_{ij}} = m_{R_{ij}} + m_{R_{ij}} (1 - m_{R_{ij}}) \ge m_{R_{ij}} \quad (2)$$

From (1) and (2), we get that

M ⊕ M ≥ M.

(ii) The ij[th] element of M⊙M is $\hat{M}_{ij} \odot \hat{M}_{ij}$ which is equal to

[$m_{L_{ij}} . m_{L_{ij}} , m_{R_{ij}} . m_{R_{ij}}$].

Since, $m_{L_{ij}}{}^2 \leq m_{L_{ij}} , m_{R_{ij}}{}^2 \leq m_{R_{ij}}$.

$\therefore$ [$m_{L_{ij}}{}^2 , m_{R_{ij}}{}^2$] which is less than or equal to [$m_{L_{ij}} , m_{R_{ij}}$].

Therefore, M ⊙ M ≤ M.

Definition: 2.4 Let M = ($\hat{M}_{ij}$) be an nxn interval fuzzy number matrix. Then M is **Nearly irreflexive** iff $\hat{M}_{ii} \leq \hat{M}_{ij}$ for all i,j=1,2,……,n.

Property: 2.5 Let M and N be two interval fuzzy number matrices, then

(i) M ⊕ N ≥ M ⊙ N
(ii) If M and N are nearly irreflexive then M ⊕ N and M ⊙ N are nearly irreflexive.

Proof: Let M = ($\hat{M}_{ij}$) where $\hat{M}_{ij}$ = [$m_{L_{ij}} , m_{R_{ij}}$] and N = ($\hat{N}_{ij}$) where

$\hat{N}_{ij}$ = [$n_{L_{ij}} , n_{R_{ij}}$].

(i) The ij[th] element of M ⊕ N is $\hat{M}_{ij} \oplus \hat{N}_{ij}$ which is equal to

[$m_{L_{ij}} + n_{L_{ij}} - m_{L_{ij}} . n_{L_{ij}} , m_{R_{ij}} + n_{R_{ij}} - m_{R_{ij}} . n_{R_{ij}}$] and that of

M ⊙ N is $\hat{M}_{ij} \odot \hat{N}_{ij}$ = [$m_{L_{ij}} . n_{L_{ij}} , m_{R_{ij}} . n_{R_{ij}}$].

Now consider $m_{L_{ij}} + n_{L_{ij}} - m_{L_{ij}} . n_{L_{ij}} - m_{L_{ij}} . n_{L_{ij}} = m_{L_{ij}} . (1- n_{L_{ij}}) + n_{L_{ij}} .(1- m_{L_{ij}}) \geq 0$ (since $0 \leq m_{L_{ij}} \leq 1$ & $0 \leq n_{L_{ij}} \leq 1$)

$\therefore m_{L_{ij}} + n_{L_{ij}} - m_{L_{ij}} . n_{L_{ij}} \geq m_{L_{ij}} . n_{L_{ij}}$.

Similarly, $m_{R_{ij}} + n_{R_{ij}} - m_{R_{ij}} . n_{R_{ij}} \geq m_{R_{ij}} . n_{R_{ij}}$

Thus, M ⊕ N ≥ M ⊙ N .

(ii) Since M and N are nearly irreflexive, $\hat{M}_{ii} \leq \hat{M}_{ij}$ and $\hat{N}_{ii} \leq \hat{N}_{ij}$ for all i,j.

$\therefore$ [$m_{L_{ii}} , m_{R_{ii}}$] ≤ [$m_{L_{ij}} , m_{R_{ij}}$]

$\therefore m_{L_{ii}} \leq m_{L_{ij}}$ and $m_{R_{ii}} \leq m_{R_{ij}}$ (1)

Similarly, [$n_{L_{ii}} , n_{R_{ii}}$] ≤ [$n_{L_{ij}} , n_{R_{ij}}$]

$\therefore n_{L_{ii}} \leq n_{L_{ij}}$ and $n_{R_{ii}} \leq n_{R_{ij}}$ (2)

Let $\hat{C}_{ij}$ and $\hat{D}_{ij}$ be the ijth elements of M ⊕ N and M ⊙ N respectively.Then

$$\hat{C}_{ij} - \hat{C}_{ii} = \left[m_{L_{ij}} + n_{L_{ij}} - m_{L_{ij}} . n_{L_{ij}} . m_{R_{ij}} + n_{R_{ij}} - m_{R_{ij}} . n_{R_{ij}} \right] - \left[m_{L_{ii}} + n_{L_{ii}} - m_{L_{ii}} . n_{L_{ii}} . m_{R_{ii}} + n_{R_{ii}} - m_{R_{ii}} . n_{R_{ii}} \right]$$

.

= [$m_{L_{ij}} + n_{L_{ij}} - m_{L_{ij}} . n_{L_{ij}} - m_{L_{ii}} - n_{L_{ii}} + m_{L_{ii}} . n_{L_{ii}}$,

$m_{R_{ij}} + n_{R_{ij}} - m_{R_{ij}} . n_{R_{ij}} - m_{R_{ii}} - n_{R_{ii}} + m_{R_{ii}} . n_{R_{ii}}$]

$= [(1- m_{L_{ii}}).(1- n_{L_{ii}}) - (1- m_{L_{ij}}).(1- n_{L_{ij}}), (1- m_{R_{ii}}).(1- n_{R_{ii}}) - (1- m_{R_{ij}}).(1- n_{R_{ij}})]$

From (1) and (2), we get

$(1- m_{L_{ii}}).(1- n_{L_{ii}}) - (1- m_{L_{ij}}).(1- n_{L_{ij}}) \geq 0$ as $1- m_{L_{ii}} \geq 1- m_{L_{ij}}$

and $1- n_{L_{ii}} \geq 1- n_{L_{ij}}$ (3)

Similarly, $(1- m_{R_{ii}}).(1- n_{R_{ii}}) - (1- m_{R_{ij}}).(1- n_{R_{ij}}) \geq 0$ (4)

From (3) and (4), we get $\hat{C}_{ij} \geq \hat{C}_{ii}$

Hence, M $\oplus$ N is nearly irreflexive.

Now consider,

$\hat{D}_{ij} - \hat{D}_{ii} = [m_{L_{ij}} . n_{L_{ij}} , m_{R_{ij}} . n_{R_{ij}}] - [m_{L_{ii}} . n_{L_{ii}} , m_{R_{ii}} . n_{R_{ii}}]$

$= [m_{L_{ij}} . n_{L_{ij}} - m_{L_{ii}} . n_{L_{ii}} , m_{R_{ij}} . n_{R_{ij}} - m_{R_{ii}} . n_{R_{ii}}]$

But $m_{L_{ij}} . n_{L_{ij}} - m_{L_{ii}} . n_{L_{ii}} \geq 0$, $m_{R_{ij}} . n_{R_{ij}} - m_{R_{ii}} . n_{R_{ii}} \geq 0$ (By (1))

$\therefore \hat{D}_{ij} \geq \hat{D}_{ii}$

Hence, M $\odot$ N is nearly irreflexive.

Property: 2.6 Let M, N and Q be three interval fuzzy number matrices. If M $\leq$ N, then
(1) M $\oplus$ Q $\leq$ N $\oplus$ Q
(2) M $\odot$ Q $\leq$ N $\odot$ Q

Proof: Let M = $(\hat{M}_{ij})$ where $\hat{M}_{ij}$ =[$m_{L_{ij}}$, $m_{R_{ij}}$] , N = $(\hat{N}_{ij})$ where $\hat{N}_{ij}$ = [$n_{L_{ij}}$, $n_{R_{ij}}$] and Q=($\hat{Q}_{ij}$) where $\hat{Q}_{ij}$ =[$q_{L_{ij}}$, $q_{R_{ij}}$].

(1) Let $\hat{D}_{ij}, \hat{E}_{ij}, \hat{F}_{ij}$ and $\hat{G}_{ij}$ be the ijth elements of M $\oplus$ Q,N $\oplus$ Q,M $\odot$ Q and N $\odot$ Q respectively.

But the ijth element of M $\oplus$ Q , N $\oplus$ Q , M $\odot$ Q and N $\odot$ Q is $\hat{M}_{ij} \oplus \hat{Q}_{ij}$,
$\hat{N}_{ij} \oplus \hat{Q}_{ij}, \hat{M}_{ij} \odot \hat{Q}_{ij}$ and $\hat{N}_{ij} \odot \hat{Q}_{ij}$ .Then

$\hat{M}_{ij} \oplus \hat{Q}_{ij} = \hat{D}_{ij} = [m_{L_{ij}} + q_{L_{ij}} - m_{L_{ij}} . q_{L_{ij}} , m_{R_{ij}} + q_{R_{ij}} - m_{R_{ij}} . q_{R_{ij}}]$

$\hat{N}_{ij} \oplus \hat{Q}_{ij} = \hat{E}_{ij} = [n_{L_{ij}} + q_{L_{ij}} - n_{L_{ij}} . q_{L_{ij}} , n_{R_{ij}} + q_{R_{ij}} - n_{R_{ij}} . q_{R_{ij}}]$

$\hat{M}_{ij} \odot \hat{Q}_{ij} = \hat{F}_{ij} = [m_{L_{ij}} . q_{L_{ij}} , m_{R_{ij}} . q_{R_{ij}}]$

$\hat{N}_{ij} \odot \hat{Q}_{ij} = \hat{G}_{ij} = [n_{L_{ij}} . q_{L_{ij}} , n_{R_{ij}} . q_{R_{ij}}]$

Since M $\leq$ N, $m_{L_{ij}} \leq n_{L_{ij}}$.

$\therefore m_{L_{ij}} (1- q_{L_{ij}}) \leq n_{L_{ij}} (1- q_{L_{ij}})$

$\therefore m_{L_{ij}} - m_{L_{ij}} . q_{L_{ij}} \leq n_{L_{ij}} - n_{L_{ij}} . q_{L_{ij}}$

i.e., $m_{L_{ij}} + q_{L_{ij}} - m_{L_{ij}} . q_{L_{ij}} \leq n_{L_{ij}} + q_{L_{ij}} - n_{L_{ij}} . q_{L_{ij}}$

Thus, we get

$$[m_{L_{ij}} + q_{L_{ij}} - m_{L_{ij}} . q_{L_{ij}} , m_{R_{ij}} + q_{R_{ij}} - m_{R_{ij}} . q_{R_{ij}}] \leq [n_{L_{ij}} + q_{L_{ij}} - n_{L_{ij}} . q_{L_{ij}} , n_{R_{ij}} + q_{R_{ij}} - n_{R_{ij}} . q_{R_{ij}}]$$

i.e., $\hat{D}_{ij} \leq \hat{E}_{ij}$ for all i,j.

$\therefore \hat{M}_{ij} \oplus \hat{Q}_{ij} \leq \hat{N}_{ij} \oplus \hat{Q}_{ij}$ for all i,j.

Hence, M $\oplus$ Q $\leq$ N $\oplus$ Q.

(2) Also $[m_{L_{ij}} . q_{L_{ij}} , m_{R_{ij}} . q_{R_{ij}}] \leq [n_{L_{ij}} . q_{L_{ij}} , n_{R_{ij}} . q_{R_{ij}}]$

i.e., $\hat{F}_{ij} \leq \hat{G}_{ij}$ for all i,j.

$\therefore \hat{M}_{ij} \odot \hat{Q}_{ij} \leq \hat{N}_{ij} \odot \hat{Q}_{ij}$ $\forall$ i,j.

Hence, M $\odot$ Q $\leq$ N $\odot$ Q.

Property: 2.7 For any two interval fuzzy number matrices M and N,
(a) M $\oplus$ N $\geq$ M $\vee$ N $\geq$ M $\ominus$ N
(b) (M $\vee$ N) $\vee$ (M $\ominus$ N) = M $\vee$ N
(c) (M $\vee$ N) $\ominus$ (M $\ominus$ N) $\leq$ N
(d) M $\oplus$ N $\geq$ (M $\vee$ N) $\vee$ (M $\ominus$ N)
(e) M $\oplus$ N $\geq$ (M $\vee$ N) $\ominus$ (M $\ominus$ N)

Proof:
(a) M $\oplus$ N $\geq$ M $\vee$ N $\geq$ M $\ominus$ N

Let $\hat{C}_{ij}, \hat{D}_{ij}$ and $\hat{E}_{ij}$ be the ijth elements of M $\oplus$ N, M $\vee$ N and M $\ominus$ N respectively.

But the ij[th] elements of M $\oplus$ N, M $\vee$ N and M $\ominus$ N are $\hat{M}_{ij} \oplus \hat{N}_{ij}$, $\hat{M}_{ij} \vee \hat{N}_{ij}$ and $\hat{M}_{ij} \ominus \hat{N}_{ij}$. Then

$$\hat{M}_{ij} \oplus \hat{N}_{ij} = \hat{C}_{ij}$$

$$= [m_{L_{ij}} + n_{L_{ij}} - m_{L_{ij}} . n_{L_{ij}} , m_{R_{ij}} + n_{R_{ij}} - m_{R_{ij}} . n_{R_{ij}}]$$

$$= \begin{cases} [m_{L_{ij}} + n_{L_{ij}}(1 - m_{L_{ij}}), m_{R_{ij}} + n_{R_{ij}}(1 - m_{R_{ij}})] \\ \geq [m_{L_{ij}}, m_{R_{ij}}] \\ \geq \hat{M}_{ij} \\ [n_{L_{ij}} + m_{L_{ij}}(1 - n_{L_{ij}}), n_{R_{ij}} + m_{R_{ij}}(1 - n_{R_{ij}})] \\ \geq [n_{L_{ij}}, n_{R_{ij}}] \\ \geq \hat{N}_{ij} \end{cases} \qquad (1)$$

$$\hat{M}_{ij} \vee \hat{N}_{ij} = \hat{D}_{ij} = \max\{ \hat{M}_{ij}, \hat{N}_{ij} \}$$

$$\leq [m_{L_{ij}} + n_{L_{ij}} - m_{L_{ij}} . n_{L_{ij}} , m_{R_{ij}} + n_{R_{ij}} - m_{R_{ij}} . n_{R_{ij}}]$$

$$= \hat{C}_{ij} \text{ (from (1))}$$

Thus, $\hat{C}_{ij} \geq \hat{D}_{ij}$ for all i,j.

i.e., $\hat{M}_{ij} \oplus \hat{N}_{ij} \geq \hat{M}_{ij} \vee \hat{N}_{ij}$ $\forall$ i,j.

Hence, M $\oplus$ N $\geq$ M $\vee$ N (2)

Now consider

$\hat{E}_{ij} = \hat{M}_{ij} \ominus \hat{N}_{ij} = [\, m_{L_{ij}} \ominus n_{L_{ij}}, m_{R_{ij}} \ominus n_{R_{ij}} \,]$

where $m_{L_{ij}} \ominus n_{L_{ij}} = \begin{cases} m_{L_{ij}}, & m_{L_{ij}} > n_{L_{ij}} \\ 0, & m_{L_{ij}} \le n_{L_{ij}} \end{cases}$

i.e., $m_{L_{ij}} \ominus n_{L_{ij}} \le m_{L_{ij}}$

$\le \max\{ m_{L_{ij}}, n_{L_{ij}} \}$

$m_{R_{ij}} \ominus n_{R_{ij}} \le m_{R_{ij}}$

$\le \max\{ m_{R_{ij}}, n_{R_{ij}} \}$

$\therefore \hat{E}_{ij} \le \max\{[\, m_{L_{ij}}, m_{R_{ij}} \,], [\, n_{L_{ij}}, n_{R_{ij}} \,]\}$

$\le \max\{ \hat{M}_{ij}, \hat{N}_{ij} \}$

$\le \hat{M}_{ij} \vee \hat{N}_{ij}$

Thus, $M \ominus N \le M \vee N$ (3)

From (2) and (3), we get $M \oplus N \ge M \vee N \ge M \ominus N$.

Let $\hat{D}_{ij}, \hat{E}_{ij}$ and $\hat{H}_{ij}$ be the ijth elements of $M \vee N$, $M \ominus N$ and $(M \vee N) \vee (M \ominus N)$ respectively.

But the ijth elements of $M \vee N$, $M \ominus N$ and $(M \vee N) \vee (M \ominus N)$ is $\hat{M}_{ij} \vee \hat{N}_{ij}$, $\hat{M}_{ij} \ominus \hat{N}_{ij}$ and $(\hat{M}_{ij} \vee \hat{N}_{ij}) \vee (\hat{M}_{ij} \ominus \hat{N}_{ij})$. Then

$\hat{D}_{ij} = \hat{M}_{ij} \vee \hat{N}_{ij} = [\, m_{L_{ij}} \vee n_{L_{ij}}, m_{R_{ij}} \vee n_{R_{ij}} \,]$ where $m_{L_{ij}} \vee n_{L_{ij}} = \max\{ m_{L_{ij}}, n_{L_{ij}} \}$

$\hat{E}_{ij} = \hat{M}_{ij} \ominus \hat{N}_{ij} = [\, m_{L_{ij}} \ominus n_{L_{ij}}, m_{R_{ij}} \ominus n_{R_{ij}} \,]$ where

$$m_{L_{ij}} \ominus n_{L_{ij}} = \begin{cases} m_{L_{ij}}, & m_{L_{ij}} > n_{L_{ij}} \\ 0, & m_{L_{ij}} \le n_{L_{ij}} \end{cases}$$

(a) $(M \vee N) \vee (M \ominus N) = M \vee N$

Case (i): $m_{L_{ij}} > n_{L_{ij}}, m_{R_{ij}} > n_{R_{ij}}$.

$\therefore \hat{M}_{ij} \ominus \hat{N}_{ij} = [\, m_{L_{ij}}, m_{R_{ij}} \,]$ and $\hat{M}_{ij} \vee \hat{N}_{ij} = [\, m_{L_{ij}}, m_{R_{ij}} \,]$. Then

$\hat{H}_{ij} = [\, m_{L_{ij}}, m_{R_{ij}} \,] \vee [\, m_{L_{ij}}, m_{R_{ij}} \,]$

$= [\, m_{L_{ij}}, m_{R_{ij}} \,]$

$= \hat{M}_{ij} \vee \hat{N}_{ij}$

$\therefore \hat{H}_{ij} = \hat{D}_{ij}$

Case (ii) : $m_{L_{ij}} > n_{L_{ij}}, m_{R_{ij}} \le n_{R_{ij}}$.

$\therefore \hat{M}_{ij} \ominus \hat{N}_{ij}$ = [$m_{L_{ij}}$,0] and $\hat{M}_{ij} \vee \hat{N}_{ij}$ = [$m_{L_{ij}}$, $n_{R_{ij}}$]. Then

$\hat{H}_{ij}$ = [$m_{L_{ij}}$,0] $\vee$ [$m_{L_{ij}}$, $n_{R_{ij}}$]

= [$m_{L_{ij}}$, $n_{R_{ij}}$]

= $\hat{M}_{ij} \vee \hat{N}_{ij}$

$\therefore \hat{H}_{ij} = \hat{D}_{ij}$

Case (iii): $m_{L_{ij}} \le n_{L_{ij}}, m_{R_{ij}} > n_{R_{ij}}$

$\therefore \hat{M}_{ij} \ominus \hat{N}_{ij}$ = [0, $m_{R_{ij}}$] and $\hat{M}_{ij} \vee \hat{N}_{ij}$ = [$n_{L_{ij}}$, $m_{R_{ij}}$]. Then

$\hat{H}_{ij}$ = [0, $m_{R_{ij}}$] $\vee$ [$n_{L_{ij}}$, $m_{R_{ij}}$]

= [$n_{L_{ij}}$, $m_{R_{ij}}$]

= $\hat{M}_{ij} \vee \hat{N}_{ij}$

= $\hat{D}_{ij}$

Case (iv): $m_{L_{ij}} \le n_{L_{ij}}, m_{R_{ij}} \le n_{R_{ij}}$

$\therefore \hat{M}_{ij} \ominus \hat{N}_{ij}$ = [0,0] and $\hat{M}_{ij} \vee \hat{N}_{ij}$ = [$n_{L_{ij}}$, $n_{R_{ij}}$]. Then

$\hat{H}_{ij}$ = [0,0] $\vee$ [$n_{L_{ij}}$, $n_{R_{ij}}$]

= [$n_{L_{ij}}$, $n_{R_{ij}}$]

= $\hat{M}_{ij} \vee \hat{N}_{ij}$

= $\hat{D}_{ij}$

$\therefore \hat{H}_{ij} = \hat{D}_{ij}$

$\therefore$ In all cases, $\hat{D}_{ij} = \hat{H}_{ij}$.

i.e., $\hat{M}_{ij} \vee \hat{N}_{ij} = (\hat{M}_{ij} \vee \hat{N}_{ij}) \vee (\hat{M}_{ij} \ominus \hat{N}_{ij})$

Hence M $\vee$ N = (M $\vee$ N) $\vee$ (M $\ominus$ N).

(c) (M $\vee$ N) $\ominus$ (M $\ominus$ N) $\le$ N

Let $\hat{F}_{ij}$ be the ijth elements of (M $\vee$ N) $\ominus$ (M $\ominus$ N).But the ijth element of (M $\vee$ N) $\ominus$ (M $\ominus$ N) is ($\hat{M}_{ij} \vee \hat{N}_{ij}$) $\ominus$ ($\hat{M}_{ij} \ominus \hat{N}_{ij}$).

Case (i): $m_{L_{ij}} > n_{L_{ij}}, m_{R_{ij}} > n_{R_{ij}}$

$\therefore \hat{M}_{ij} \ominus \hat{N}_{ij}$ = [$m_{L_{ij}}$, $m_{R_{ij}}$] and $\hat{M}_{ij} \vee \hat{N}_{ij}$ = [$m_{L_{ij}}$, $m_{R_{ij}}$].Then

$\hat{F}_{ij}$ = [$m_{L_{ij}}$, $m_{R_{ij}}$] $\ominus$ [$m_{L_{ij}}$, $m_{R_{ij}}$]

= $[m_{L_{ij}} \ominus m_{L_{ij}}, m_{R_{ij}} \ominus m_{R_{ij}}]$
= [0,0].

Case (ii): $m_{L_{ij}} \leq n_{L_{ij}}, m_{R_{ij}} > n_{R_{ij}}$

$\therefore \hat{M}_{ij} \ominus \hat{N}_{ij}$ = $[0, m_{R_{ij}}]$ and $\hat{M}_{ij} \vee \hat{N}_{ij} = [n_{L_{ij}}, m_{R_{ij}}]$.Then

$\hat{F}_{ij} = [0, m_{R_{ij}}] \ominus [n_{L_{ij}}, m_{R_{ij}}]$

$= [0 \ominus n_{L_{ij}}, m_{R_{ij}} \ominus m_{R_{ij}}]$

$= [n_{L_{ij}}, 0]$.

Case (iii) : $m_{L_{ij}} > n_{L_{ij}}, m_{R_{ij}} \leq n_{R_{ij}}$

$\therefore \hat{M}_{ij} \ominus \hat{N}_{ij} = [m_{L_{ij}}, 0]$ and $\hat{M}_{ij} \vee \hat{N}_{ij} = [m_{L_{ij}}, n_{R_{ij}}]$.Then

$\hat{F}_{ij} = [m_{L_{ij}}, 0] \ominus [m_{L_{ij}}, n_{R_{ij}}]$

$= [m_{L_{ij}} \ominus m_{L_{ij}}, 0 \ominus n_{R_{ij}}]$

$= [0, n_{R_{ij}}]$

Case (iv): $m_{L_{ij}} \leq n_{L_{ij}}, m_{R_{ij}} \leq n_{R_{ij}}$

$\therefore \hat{M}_{ij} \ominus \hat{N}_{ij}$ = [0,0] and $\hat{M}_{ij} \vee \hat{N}_{ij} = [n_{L_{ij}}, n_{R_{ij}}]$. Then

$\hat{F}_{ij} = [0,0] \ominus [n_{L_{ij}}, n_{R_{ij}}]$

$= [0 \ominus n_{L_{ij}}, 0 \ominus n_{R_{ij}}]$

$= [n_{L_{ij}}, n_{R_{ij}}]$

$= \hat{N}_{ij}$

From case (i) to case (iv), we see that the ijth element of $\hat{F}_{ij}$ is either 0 or $\hat{N}_{ij}$ according as $[m_{L_{ij}}, m_{R_{ij}}]$ > or $\leq$ to $[n_{L_{ij}}, n_{R_{ij}}]$.

Thus, $(M \vee N) \ominus (M \ominus N) \leq N$.

(d) $M \oplus N \geq (M \vee N) \vee (M \ominus N)$

$(M \vee N) \vee (M \ominus N) = M \vee N \leq M \oplus N$ (By Property 2.6)

Hence, $M \oplus N \geq (M \vee N) \vee (M \ominus N)$.

(e) $M \oplus N \geq (M \vee N) \ominus (M \ominus N)$

It is obvious that $N \leq M \vee N$

$\therefore (M \vee N) \ominus (M \ominus N) \leq N$ (by (ii))
$\leq M \vee N$
$\leq M \oplus N$

Hence, $M \oplus N \geq (M \vee N) \ominus (M \ominus N)$.

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