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On K-torse-forming vector field in a trans-Sasakian generalized Sasakian space-form

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ABSTRACT

The purpose of the present paper is to study K-torse-forming vector fields in trans-Sasakian generalized Sasakian space-forms. We prove the condition for Ricci tensor S to be semiconjugated with the characteristic vector field ξ which is K-torse-forming.

Keywords: *K-Torseforming vector field, generalized Sasakian space- form, trans-Sasakian, contact transformation.*

Mathematics Subject Classification[2010]: *53D10.*

1. INTRODUCTION

Torse foming vector fields were introduced by K.Yano [9] in 1944 and the complex analogue of a torse forming vector field was introduced by S.Yamaguchi [8] in 1979. This vector field is known as a Kahlerian torse forming vector field or simply a K-torse-forming vector field. P. Alegre , D. E. Blair and A. Carriazo [1]introduced the concept of generalized Sasakian space-forms and proved some classification results. Further the behavior of such spaces under D-conformal transformations are studied by P. Alegre and A. Carriazo [2]. In this paper we study the generalized Sasakian space-forms admitting a K-torse forming vector field. In section 2, we give a breif review of basic results. Section 3 is devoted to semiconjugacy of the Ricci tensor S with K-torse forming vector field ξ. In section 4, we consider infinitesimal contact transformation and prove conditions for the transformation to be a strict contact transformation.

2. PRELIMINARIES

An odd dimensional Riemannian manifold (M, g) is called an almost contact metric manifold if there exist on M, a (1,1) tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \qquad g(X,\xi) = \eta(X),$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.1}$$

As a consequence, we obtain

$$\eta(\phi X) = 0, \qquad \phi\xi = 0, \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad (\nabla_X\eta)(Y) = g(\nabla_X\xi, Y), \tag{2.3}$$

for any vector fields X,Y on M.

An almost contact metric manifold is a Sasakian manifold if

$$(\nabla_X\phi)(Y) = g(X,Y)\xi - \eta(Y)X. \tag{2.4}$$

An almost contact metric manifold (M, ϕ, ξ, η, g) is called a trans-Sasakian manifold [6] if there exist two functions α and β on M such that

$$(\nabla_X\phi)(Y) = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.5}$$

for any vector fields X,Y on M.

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From (2.5), it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \quad (2.6)$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \quad (2.7)$$

From the well known Oubina's result[6]: for dimensions over or equal to 5 there exist $(\alpha, 0)$ and $(0, \beta)$ trans-Sasakian manifolds only. P. Alegre, D. Blair and A. Carriazo [1] introduced and studied generalized Sasakian space -forms.

An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be a generalized Sasakian space-form if there exist differentiable functions f_1, f_2 and f_3 on M such that the curvature tensor R of M satisfies

$$R(X,Y)Z = f_1 R_1(X,Y)Z + f_2 R_2(X,Y)Z + f_3 R_3(X,Y)Z, \quad (2.8)$$

for any vector fields X,Y,Z on M, where

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \quad (2.9)$$

$$R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z \quad (2.10)$$

and

$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi. \quad (2.11)$$

Throughout this paper $M(f_1, f_2, f_3)$ will denote a generelized Sasakian space-form.

In a generalized Sasakian space-form the following hold:

$$R(X,Y)\xi = (f_1 - f_3)(\eta(Y)X - \eta(X)Y), \quad (2.12)$$

$$S(Y,Z) = [(n-1)f_1 + 3f_2 - f_3]g(Y,Z) - [3f_2 + (n-2)f_3]\eta(Y)\eta(Z), \quad (2.13)$$

$$QY = [(n-1)f_1 + 3f_2 - f_3]Y - [3f_2 + (n-2)f_3]\eta(Y)\xi, \quad (2.14)$$

$$S(Y,\xi) = (n-1)(f_1 - f_3)\eta(Y), \quad (2.15)$$

$$r = n(n-1)f_1 + 3(n-1)f_2 - 2(n-1)f_3. \quad (2.16)$$

In the following we give the definitions of torse-forming vector field and K-torse forming vector fields [8].

Definition 1: *A vector field ρ defned by $g(X,\rho) = \omega(X)$ for any vector field X is said to be a torse forming vector field if*

$$(\nabla_X \omega)Y = a g(X,Y) + \pi(X)\omega(Y), \quad (2.17)$$

where a is a non zero scalar and π is a non zero 1-form.

Definition 2: *A vector field ρ is said to be K-torse-forming if*

$$\nabla_X \rho = aX + b\phi X + B(X)\rho + D(X)\phi\rho \quad (2.18)$$

or

$$(\nabla_X \omega)Z = a g(X,Z) + b g(\phi X, Z) + B(X)\omega(Z) - D(X)\omega(\phi Z), \quad (2.19)$$

where $g(X,\rho) = \omega(X)$, a and b are functions and B(X) and D(X) are 1-forms.

The functions a and b are called associated functions and the 1-forms B and D are called associate forms of ρ. Moreover if the associated functions a and b satisfy $a^2 + b^2 \neq 0$, then ρ is called a proper K-torse-forming vector field.

Remark 1: *From (2.6), it follows that in a trans-Sasakian manifold, ξ is always a K-torse-forming vector field with $a = \beta$, $b = -\alpha$, $B(X) = -\beta\eta(X)$ and $D(X) = 0$.*

Definition 3: *The tensor field T is semi-conjugated with the vector field ρ, if*

$$R(X,\rho).T = 0 \quad (2.20)$$

3. K-TORSE FORMING VECTOR FIELDS

In this section we will consider a unit K-torse forming vector field ρ in a generalized Sasakian space-form $M(f_1, f_2, f_3)$.

Taking $Z = \rho$ in (2.19), we have

$$B(X) = -[b\omega(\phi X) + a\omega(X)]. \tag{3.1}$$

Taking $Z = \phi\rho$ in (2.19) and using (2.1), we obtain

$$D(X) = \frac{a[(\omega(X) - \omega(\phi X)]}{2[1 - (\eta(\rho))^2]}. \tag{3.2}$$

Plugging (3.1) and (3.2) in (2.19), we have

$$\begin{aligned}(\nabla_X\omega)Z &= a[g(X,Z) - \omega(X)\omega(Z)] + b[g(\phi X, Z) - \omega(\phi X)\omega(Z)] \\ &-\lambda a[\omega(X)\omega(\phi Z) - \omega(\phi X))\omega(\phi Z)],\end{aligned} \tag{3.3}$$

where $\lambda = \frac{1}{2(1-(\eta(\rho))^2)}$.

Using (3.3) and (2.4) in the Ricci identity and taking $Z = \xi$ in the resultant expression, we get

$$\begin{aligned}-\omega(R(X,Y)\xi) &= (Xa)[\eta(Y) - \eta(\rho)\omega(Y)] - (Ya)[\eta(X) - \eta(\rho)\omega(X)] \\ &+(\lambda a(\eta(\rho(a - b + \lambda a) - 1)))[\omega(\phi Y)\omega(X) - \omega(\phi X)\omega(Y)] \\ &-(a^2 + \eta(\rho)(b + \lambda b\eta(\rho) + \lambda a))[\eta(X)\omega(Y) - \eta(Y)\omega(X)] \\ &+(a(b - \lambda\eta(\rho)(b\eta(\rho) - 1)))[\eta(Y)\omega(\phi X) - \eta(X)\omega(\phi Y)] \\ &+\eta(\rho)[(Yb)\omega(\phi X) - (Xb)\omega(\phi Y)].\end{aligned} \tag{3.4}$$

Putting $X = \rho$ in (3.4) and using (2.8), we obtain

$$(\rho a) + a^2 + b\eta(\rho) + \lambda ab(\eta(\rho))^2 + \lambda a\eta(\rho) = f_3 - f_1$$

and

$$[-\lambda a^2\eta(\rho) - \lambda ab\eta(\rho) + \lambda^2 a^2\eta(\rho) - \lambda a - ab\eta(\rho) + \lambda ab(\eta(\rho))^3 + \lambda a(\eta(\rho))^2] = 0.$$

If ρ is orthogonal to ξ then the above equations reduce to

$(\rho a) + a^2 = f_3 - f_1$ and $\lambda a = 0$.

Since $\lambda \neq 0$, the second equation implies $a = 0$. This with first equation gives $f_1 = f_3$.

Thus we have

Theorem 2: *If a torse forming vector field ρ in a generalized Sasakian space-form is orthogonal to ξ then we have $f_1 = f_3$.*

Since the characteristic vector field ξ is a K-torse forming vector field in $M(f_1, f_2, f_3)$, where M is a trans-Sasakian manifold, by remark 1, we obtain $a^2 + b^2 = \alpha^2 + \beta^2$ and hence $a^2 + b^2 \neq 0$ provided $(\alpha, \beta) \neq (0,0)$.

Thus we have

Theorem 3: *In a non co-symplectic trans-Sasakian manifold of dimension $n \geq 5$ the K-torse-forming vector field ξ is proper.*

Using the definition of Riemannian curvature tensor and by remark 1, we have

$$\begin{aligned}R(X,Y)\xi &= (Xa)[Y - \eta(Y)\xi] - (Ya)[X - \eta(X)\xi] + a[-(\nabla_X\eta)Y + (\nabla_Y\eta)X]\xi \\ &+a[-(\nabla_X\xi)\eta(Y) + (\nabla_Y\xi)\eta(X)] + (Xb)\phi Y - (Yb)\phi X + b[(\nabla_X\phi)Y - (\nabla_Y\phi)X].\end{aligned} \tag{3.5}$$

From (2.5), (2.6) and (2.7) in (3.5), we get

$$\begin{aligned}R(X,Y)\xi &= -(Xa)\phi^2 Y + (Ya)\phi^2 X + (Xb)\phi Y - (Yb)\phi X + 2[a\alpha + \beta b]g(\phi X, Y)\xi \\ &+[(a)^2 + b(\alpha + \beta)][\eta(X)Y - \eta(Y)X] + ab[\eta(X)\phi Y - \eta(Y)\phi X].\end{aligned} \tag{3.6}$$

From (3.6), we have

$$S(X,\xi) = (2-n)(X\alpha) - (\xi\alpha) + (\phi X)b - (n-1)[\alpha^2 + b(\alpha+\beta)]\eta(X). \tag{3.7}$$

Suppose the Ricci tensor S is semi-conjugated with the K-torse-forming vector field ξ. i.e. $R(X,\xi).S(Y,Z) = 0$. Then we have

$$S(R(X,\xi)Y,Z) + S(Y,R(X,\xi)Z) = 0.$$

Putting Z=ξ in the above equation, we obtain

$$S(R(X,\xi)Y,\xi) + S(Y,R(X,\xi)\xi) = 0. \tag{3.8}$$

For constants α and b, (3.6) reduces to

$$\begin{aligned} R(X,Y)\xi = \; & 2[a\alpha + \beta b]g(\phi X,Y)\xi + [(\alpha)^2 + b(\alpha+\beta)][\eta(X)Y - \eta(Y)X] \\ & +\alpha b[\eta(X)\phi Y - \eta(Y)\phi X]. \end{aligned}$$

From the above equation, we have

$$R(X,\xi)Y = \; A\eta(X)\phi Y + B(\eta(Y)X - g(X,Y)\xi) + C(g(\phi X,Y)\xi - \eta(Y)\phi X), \tag{3.9}$$

where $A = -2(a\alpha + b\beta), B = -(\alpha^2 + b(\alpha+\beta))$ and $C = -\alpha b$.

Using (3.9) in (3.8), we have

$$\begin{aligned} R(X,\xi).S(Y,Z) = \; & B(\eta(Y)S(X,Z) - g(X,Y)S(\xi,Z) + \eta(Z)S(X,Y) - g(X,Z)S(\xi,Y)) \\ & +C(g(\phi X,Y)S(\xi,Z) - \eta(Y)S(\phi X,Z) + g(\phi X,Z)S(\xi,Y) - \eta(Z)S(\phi X,Y)). \end{aligned} \tag{3.10}$$

Using (2.13) and (2.15) in (3.10), we get

$$\begin{aligned} R(X,\xi).S(Y,Z) = \; & B[-(n-1)(f_1 - f_3)(g(X,Y)\eta(Z) + g(X,Z)\eta(Y)) \\ & +D(\eta(Y)g(X,Z) + \eta(Z)g(X,Y)) - 2E\eta(X)\eta(Y)\eta(Z)] \\ & +C[(n-1)(f_1 - f_3)(\eta(Z)g(\phi X,Y) + g(\phi X,Z)\eta(Y)) \\ & -D(\eta(Y)g(\phi X,Z) + \eta(Z)g(\phi X,Y))], \end{aligned} \tag{3.11}$$

where $D = (n-1)f_1 + 3f_2 - f_3$ and $E = 3f_2 + (n-2)f_3$.

If $E = 0$ then $D = (n-1)(f_1 - f_3)$ and consequently we have $R(X,\xi).S = 0$.

Conversely, suppose $R(X,\xi).S = 0$.

Then from (3.11), we have

$$\begin{aligned} & B[(D-(n-1)(f_1 - f_3))(g(X,Y)\eta(Z) + g(X,Z)\eta(Y)) - 2E\eta(X)\eta(Y)\eta(Z)] \\ & +C[((n-1)(f_1 - f_2) - D)(\eta(Z)g(\phi X,Y) + g(\phi X,Z)\eta(Y))] = 0. \end{aligned} \tag{3.12}$$

The above equation implies

$$\begin{aligned} & E[B(g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)) \\ & +C(\eta(Z)g(\phi X,Y) + g(\phi X,Z)\eta(Y))] = 0. \end{aligned} \tag{3.13}$$

Then either $E = 0$ or

$$\begin{aligned} & [B(g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)) \\ & +C(\eta(Z)g(\phi X,Y) + g(\phi X,Z)\eta(Y))] = 0. \end{aligned}$$

Taking $Y = \xi$ in the second equation, we get

$$B(g(\phi X,\phi Z)) + Cg(\phi X,Z) = 0.$$

Taking $X = Z = e_i$, where $\{e_i\}, i = 1,\dots,n$ is an orthonormal basis of $T_x(M)$ at each point $x \in M$ and taking the summation over $i = 1,\dots,n$, we obtain $B = 0$.

But from (2.12) and (3.6) with a and b as constants, we have

$$\begin{aligned}&((f_1 - f_3) - ((a)^2 + b(\alpha+\beta))(\eta(X)g(Y,W) - \eta(Y)g(X,W))\\ &= 2[a\alpha + \beta b]g(\phi X, Y)\eta(W) + ab(\eta(X)g(\phi Y, W) - \eta(Y)g(\phi X, W)).\end{aligned} \tag{3.14}$$

Taking $Y = W = e_i$ and taking summation over $\{e_i\}, i = 1, \dots, n$, we have

$$f_1 - f_3 = a^2 + b(\alpha+\beta) = -B.$$

From theorem 4.2 of [2] for an α-Sasakian manifold or a co-symplectic manifold, we have

$$f_1 - \alpha^2 = f_2 = f_3.$$

The above equation with $3f_2 + (n-2)f_3 = 0$ implies either $f_2 = 0$ (holds on 3-dimensional manifolds) or $n = -1$(not possible).

From the above discussion, we conclude that

Theorem 4: *In a trans-Sasakian generalized Sasakian space-form of dimension 5 or more $f_1 \neq f_3$, the Ricci tensor S is semiconjugated with the K-torseforming vector field ξ if and only if* $3f_2 + (n-2)f_3 = 0$.

Since ξ is a non-zero vector field, from theorem 2, it follows that $f_1 \neq f_3$.

Combining theorem 2 and 4, we can state that

Theorem 5: *In a $(0,\beta)$-trans-Sasakian generalized Sasakian space-form of dimension ≥ 5, the Ricci tensor S is semi-conjugated with the K-torse-forming vector field ξ if and only if* $3f_2 + (n-2)f_3 = 0$.

4. INFINITESIMAL CONTACT TRANSFORMATION.

Definition 4: *A vector field V on a contact manifold with contact form η is said to be an infinitesimal contact transformation if V satisfies*

$$(L_V\eta)X = \sigma\eta(X) \tag{4.1}$$

for a scalar function σ where L_V denotes the lie differentiation with respect to V. Especially, if σ vanishes identically, then it is called an infinitesimal strict contact transformation.

Let us now suppose that in a generalized Sasakian space-form, the infinitesimal contact transformation leaves the Ricci tensor invariant,then we have

$$(L_VS)(X,Y) = 0, \tag{4.2}$$

which gives

$$(L_VS)(X,\xi) = 0. \tag{4.3}$$

On the other hand, we have

$$(L_VS)(X,\xi) = L_VS(X,\xi) - S(L_VX,\xi) - S(X,L_V\xi). \tag{4.4}$$

By virtue of (3.7) and (4.3), the equation (4.4) yields

$$0 = (n-1)L_V[a^2 + b\alpha + b\beta]\eta(X) + (n-1)[a^2 + b\alpha + b\beta](L_V\eta)X - S(X, L_V\xi). \tag{4.5}$$

Putting $X = \xi$ in (4.5) , using (3.7), we obtain

$$\eta(L_V\xi) = \sigma + \frac{bL_V[\alpha+\beta]}{[a^2 + b(\alpha+\beta)]}. \tag{4.6}$$

Taking $X = \xi$ in (4.1), we have

$$L_V\eta(\xi) + \eta(L_V\xi) = \sigma. \tag{4.7}$$

From (4.6)and (4.7), we have

$$\sigma = -\frac{bL_V[\alpha+\beta]}{2[a^2 + b(\alpha+\beta)]}. \tag{4.8}$$

Since $a = \beta, b = -\alpha$ and $\alpha + \beta = a - b$ is a constant, from (4.7), we have $\sigma = 0$.

Thus we can state that

Theorem 6: *In a trans-Sasakian generalized Sasakian space-form, if* ξ *is a K-torse-forming vector field with* a *and* b *as constants, then the infinitesimal contact transformation which leaves the Ricci tensor invariant is an infinitesimal strict contact transformation.*

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