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A NEW CLASS OF NEARLY OPEN SETS

A. Robert[1*] & S. Pious Missier[2]

[1]Deparment of Mathematics, Aditanar College, Tiruchendur, India
[2]Deparment of Mathematics, V.O. Chidambaram College, Thoothukudi, India

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ABSTRACT

In this paper we introduce a new class of sets, namely semi-open sets, using the generalized closure operator due to Dunham. We give a characterization of semi*-open sets. We also define semi*-interior point and the semi*-interior of a subset. Further we investigate fundamental properties of semi*-open sets.*

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1. INTRODUCTION

In 1963 Levine [5] introduced semi-open sets in topological spaces. After Levine's work, many mathematicians turned their attention to generalizing various concepts in topology by considering semi-open sets instead of open sets. Levine [6] defined and studied generalized closed sets in 1970. Das [2] defined semi-interior point and semi-limit point of a subset. Dunham [3] introduced the concept of generalized closure using Levine's generalized closed sets and defined a new topology τ^* and studied some of their properties.

In this paper, in line with Levine's semi-open sets, we define a new class of sets, namely semi*-open sets, using the generalized closure operator *Cl** due to Dunham. We further show that the class of semi*-open sets is placed between the class of semi-open sets due to Levine and the class of open sets. We give a characterization of semi*-open sets. We investigate fundamental properties of semi*-open sets. We also define semi*- interior point and semi*-interior of a subset. We also study some properties of semi*-interior.

2. PRELIMINARIES

Throughout this paper, (X, τ) will always denote a topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of the space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A respectively.

Definition 2.1: A subset A of a topological space (X, τ) is ***semi-open*** [5] if there is an open set U in X such that $U\subseteq A\subseteq Cl(U)$ or equivalently if $A\subseteq Cl(Int(A))$.
The class of all semi-open sets in (X, τ) is denoted by $SO(X, \tau)$.

Definition 2.2: A subset A of a topological space (X, τ) is ***pre-open*** [7] (resp. ***α-open*** [8]) if $A\subseteq Int(Cl(A))$ (resp. $A\subseteq Int(Cl(Int(A)))$).

Definition 2.3: If A is a subset of a space X, the ***semi-interior*** of A is defined as the union of all semi-open sets of X contained in A. It is denoted by ***sInt*(A)**.

Definition 2.4: A set A is called ***pointwise dense*** if $A= \cup\{Cl(\{x\}) : x\in A \text{ and } \{x\} \text{ is open}\}$.

Definition 2.5: A subset A of a space X is ***generalized closed*** (briefly g-closed) [6] if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in X.

Definition 2.6: If A is a subset of a space X, the ***generalized closure*** [3] of A is defined as the intersection of all g-closed sets in X containing A and is denoted by $Cl^*(A)$.

Corresponding author: *A. Robert[1*]*
[1]Deparment of Mathematics, Aditanar College, Tiruchendur, India

Definition 2.7: A topological space X is $\mathbf{T_{1/2}}$ [6] if every g-closed set in X is closed.

Theorem 2.8[3]**:** *Cl** is a Kuratowski closure operator in X.

Definition 2.9[3]**:** If (X, τ) is a topological space, let τ* be the topology on X defined by the closure operator *Cl** .That is, τ*= {U⊆X: *Cl**(X\U) =X\U}.

Theorem 2.10[3]**:** If (X, τ) is a topological space, then (X, τ*) is $T_{1/2}$.

Definition 2.11: A space X is ***locally indiscrete*** [9] if every open set in X is closed.

Definition 2.12: The topology on the set of integers generated by the set S of all triplets of the form {2n-1, 2n, 2n+1} as sub base is called the ***Khalimsky topology*** [4] or ***digital topology*** and it is denoted by **κ**. The collection S∪{{2n+1}: n∈ℤ} is a base for the topology κ. The digital line equipped with the Khalimsky topology is called the ***Khalimsky line or digital line***. The topological product of two Khalimsky lines (ℤ, κ) is called the Khalimsky ***plane*** or ***digital plane*** and is denoted by ($\mathbb{Z}^2$, κ^2).

3. SEMI*-OPEN SETS

Definition 3.1: A subset A of a topological space (X, τ) is called a ***semi*-open set*** if there is an open set U in X such that U⊆A⊆*Cl**(U).

Notation: The set of all semi*-open sets in (X, τ) is denoted by S*O(X, τ) or simply S*O(X).

Definition 3.2: The ***semi*-interior*** of A is defined as the union of all semi*-open sets of X contained in A. It is denoted by ***s*Int***(A).

Definition 3.3: Let A be a subset of X. A point x in X is called a ***semi*-interior point*** of A if A contains a semi*-open set containing x.

Theorem 3.4: A subset A of X is semi*-open if and only if A⊆*Cl**(*Int*(A)).

Proof: Necessity. If A is semi*-open, then there is an open set U such that U⊆A⊆*Cl**(U). Now U⊆A ⟹ U=*Int*(U)⊆*Int*(A) ⟹ A⊆*Cl**(U)⊆*Cl**(*Int*(A)).

Sufficiency. Assume that A⊆*Cl**(*Int*(A)).Take U=*Int*(A). Then U is an open set in X such that U⊆A⊆*Cl**(U). Therefore A is semi*-open.

Remark 3.5:
(i) In any space (X, τ), ϕ and X are semi*-open sets. Every nonempty semi*-open set must contain at least one nonempty open set and hence cannot be nowhere dense.

(ii) In any topological space, a singleton set is semi*-open if and only if it is open and hence a subset A of X is pointwise dense if and only if A= ∪{*Cl*({x}) : x∈A and {x} is semi*-open}.

Theorem 3.6: If $\{A_\alpha\}$ is a collection of semi*-open sets in X, then $\cup A_\alpha$ is also semi*-open in X.

Proof: Since A_α is semi*-open for each α, there is an open set U_α in X such that $U_\alpha \subseteq A_\alpha \subseteq Cl^*(U_\alpha)$.Then $\cup U_\alpha \subseteq \cup A_\alpha \subseteq \cup Cl^*(U_\alpha) \subseteq Cl^*(\cup U_\alpha)$. Since $\cup U_\alpha$ is open, $\cup A_\alpha$ is semi*-open.

Remark 3.7: The intersection of two semi*-open sets need not be semi*-open as seen from the following examples. But the intersection of a semi*-open set and an open set is semi*-open as shown in Theorem 3.10.

Example 3.8: Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X}. In the space (X, τ), the subsets A = {a, d} and B = {b, d} are semi*-open but A∩B = {d} is not semi*-open.

Example 3.9: Consider the subspace (X, τ) of the digital plane where X= {1, 2} × {1, 2, 3}.
In (X, τ), the subsets A={(1,1), (2,2)} and B={(1,3), (2,2)} are semi*-open but A∩B={(2,2)} is not semi*-open.

Theorem 3.10: If A is semi*-open in X and B is open in X, then A∩B is semi*-open in X.

Proof: Since A is semi*-open in X, there is an open set U such that $U\subseteq A\subseteq Cl^*(U)$. Since B is open, we have $U\cap B\subseteq A\cap B\subseteq Cl^*(U)\cap B\subseteq Cl^*(U\cap B)$. Hence $A\cap B$ is semi*-open in X.

Theorem 3.11: A subset A of X is semi*-open if and only if A contains a semi*-open set about each of its points.

Proof: ***Necessity:*** Obvious.

***Sufficiency*:** Let $x\in A$. Then by assumption, there is a semi*-open set U_x containing x such that $U_x \subseteq A$. Then we have $\cup\{U_x: x\in A\}=A$. By using Theorem 3.6,

A is semi*-open.

Theorem 3.12: $S^*O(X, \tau)$ forms a topology on X if and only if it is closed under finite intersection.

Proof: Follows from Remark 3.5(i) and Theorem 3.6.

Theorem 3.13: If A is any subset of X, $s^*Int(A)$ is semi*-open. In fact $s^*Int(A)$ is the largest semi*-open set contained in A.

Proof: Follows from Definition 3.2 and Theorem 3.6.

Theorem 3.14: A subset A of X is semi*-open if and only if $s^*int(A)=A$.

Proof: A is semi*-open implies $s^*Int(A)=A$ is obvious. On the other hand let $s^*Int(A)=A$.

By Theorem 3.13, $s^*Int(A)$ is semi*-open and hence A is semi*-open.

Theorem 3.15: If A is a subset of X, then $s^*Int(A)$ is the set of all semi*-interior points of A.

Proof: $x\in s^*Int(A)$ if and only if x belongs to some semi*-open subset U of A. That is, if and only if x is a semi*-interior point of A.

Corollary 3.16: A subset A of X is semi*-open if and only if every point of A is a semi*-interior point of A.

Proof: Follows from Theorem 3.14 and Theorem 3.15.

Theorem 3.17: Every open set is semi*-open.

Proof: Let U be open in X. Then $Int(U)=U$. Therefore $U\subseteq Cl^*(U)=Cl^*(Int(U))$. Hence by Theorem 3.4, U is semi*-open.

Corollary 3.18: If a subset A is semi*-open and U is open, then $A\cup U$ is semi*-open.

Proof: Follows from Theorem 3.17 and Theorem 3.6.

Remark 3.19: The converse of Theorem 3.17 is not true as shown in the following examples.

Example 3.20: Consider the topological space (X, τ) in Example 3.8. The subsets $\{a, d\}$, $\{b, d\}$ and $\{a, b, d\}$ are semi*-open in X but not open.

Example 3.21: Consider the subspace (X, τ) of the digital plane given in Example 3.9. In (X, τ), the subsets $\{(1,1),(1,3),(2,2)\}$, $\{(1,1),(1,3),(2,1),(2,2)\}$ and $\{(1,1),(1,2),(1,3),(2,2),(2,3)\}$ are semi*-open but not open.

Definition 3.22: For a topological space (X, τ), let
$\tau_{s^*}= \{U\in S^*O(X, \tau) : U\cap A\in S^*O(X, \tau) \text{ for all } A\in S^*O(X, \tau)\}$.

Theorem 3.23: If (X, τ) is a topological space, then τ_{s^*} is a topology on X finer than τ.

Proof: Clearly $\phi, X\in\tau_{s^*}$. Let $U_\alpha\in \tau_{s^*}$ and $U=\cup U_\alpha$. Since $U_\alpha\in S^*O(X, \tau)$, by using Theorem 3.6, $U\in S^*O(X, \tau)$.

Let $A\in S^*O(X, \tau)$. Then $U_\alpha\cap A\in S^*O(X, \tau)$, for each α and hence by Theorem3.6, $U\cap A=(\cup U_\alpha)\cap A =\cup(U_\alpha\cap A)\in S^*O(X, \tau)$. Therefore $U\in\tau_{s^*}$. Now let $U_1, U_2, \ldots,U_n\in\tau_{s^*}$. Then $U_1, U_2, \ldots,U_n\in S^*O(X, \tau)$ and by definition of τ_{s^*}, we get $\bigcap_{i=1}^{n} U_i \in S^*O(X, \tau)$. If $A \in S^*O(X, \tau)$, then by repeated application of the condition, we have $(\bigcap_{i=1}^{n} U_i) \cap A \in S^*O(X, \tau)$.

Hence $\bigcap_{i=1}^{n} U_i \in \tau_{s^*}$. This shows that τ_{s^*} is a topology on X. Let $V \in \tau$. By using Theorem 3.17, $V \in S^*O(X, \tau)$. Also by Theorem 3.10, $V \cap A \in S^*O(X, \tau)$ for all $A \in S^*O(X, \tau)$. Hence $V \in \tau_{s^*}$. Thus τ_{s^*} is finer than τ.

Theorem 3.24: Every semi*-open set is semi-open.

Proof: Let A be a semi*-open set. Then there is an open set U in X such that $U \subseteq A \subseteq Cl^*(U)$.

Note that $Cl^*(U) \subseteq Cl(U)$. Therefore $U \subseteq A \subseteq Cl(U)$. Hence A is semi-open.

Remark 3.25: The converse of Theorem 3.24 is not true as shown in the following examples.

Example 3.26: Consider the topological space (X, τ) given in Example 3.8. The subsets {a, c, d} and {b, c, d} are semi-open in X but not semi*-open.

Example 3.27: Consider the subspace (X, τ) of the digital plane where X= {0, 1} × {1, 2, 3}.

In (X, τ), the subsets {(1,1),(1,2)}, {(0,2),(1,1),(1,2) } and {(0,3),(1,2),(1,3)} are semi-open but not semi*-open.

Theorem 3.28: In any topological space (X, τ), $\tau \subseteq S^*O(X, \tau) \subseteq SO(X, \tau)$. That is, the class of semi*-open sets is placed between the class of open sets and the class of semi-open sets.

Proof: Follows from Theorem 3.17 and Theorem 3.24.

Remark 3.29:

(i) If (X, τ) is a locally indiscrete space, then $\tau = S^*O(X, \tau) = SO(X, \tau)$.

(ii) In the Sierpinski space (X, τ), where X = {0, 1} and $\tau = \{\phi, \{1\}, X\}$, $\tau = S^*O(X, \tau) = SO(X, \tau)$.

(iii) The inclusions in Theorem 3.28 may be strict and equality may also hold. This can be seen from the following examples.

Example 3.30: In the space (X, τ) where X= {a, b, c, d} and $\tau = \{\phi, \{a\}, \{b, c, d\}, X\}$

$SO(X, \tau) = S^*O(X, \tau) = \{\phi, \{a\}, \{b, c, d\}, X\}$.

Here $\tau = S^*O(X, \tau) = SO(X, \tau)$.

Example3.31: In the space (X, τ) where X = {a, b, c, d} and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$,

$SO(X, \tau) = S^*O(X, \tau) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$.

Here $\tau \subsetneq S^*O(X, \tau) = SO(X, \tau)$.

Example 3.32: In the space (X, τ) where X= {a, b, c, d} and $\tau = \{\phi, \{a, b\}, X\}$,

$SO(X, \tau) = \{\phi, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$; $S^*O(X, \tau) = \{\phi, \{a, b\}, X\}$.

Here $\tau = S^*O(X, \tau) \subsetneq SO(X, \tau)$.

Example 3.33: Consider the space (X, τ) where X= {a, b, c, d} and $\tau = \{\phi, \{a\}, \{a, b, c\}, X\}$.

$SO(X, \tau) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$.
$S^*O(X, \tau) = \{\phi, \{a\}, \{a, d\}, \{a, b, c\}, X\}$. Here $\tau \subsetneq S^*O(X, \tau) \subsetneq SO(X, \tau)$.

Example 3.34: Consider the subspace (X, τ) of the digital plane where X= {1, 2, 3} × {0, 1}.

If a, b, c, d, e, f denote the points (1,0), (1,1), (2, 0), (2,1), (3,0), (3,1) respectively, then

τ={ϕ,{b},{f},{a,b},{b,f},{e,f},{a,b,f},{b,d,f},{b,e,f},{a,b,d,f},{a,b,e,f},{b,d,e,f},{a,b,d,e,f},X}.

SO(X)={ϕ,{b},{f},{a,b},{b,c},{b,d},{b,f},{c,f},{d,f},{e,f},{a,b,c},{a,b,d},{a,b,f},{b,c,d},{b,c,f},{b,d,f},{b,e,f},{c,d,f},{c,e,f},{d,e,f},{a,b,c,d},{a,b,c,f},{a,b,d,f},{a,b,e,f},{b,c,d,f},{b,c,e,f},{b,d,e,f},{c,d,e,f},{a,b,c,d,f},{a,b,c,e,f},{a,b,d,e,f},{b,c,d,e,f},X}.

S*O(X)={ϕ,{b},{f},{a,b},{b,c},{b,f},{c,f},{e,f},{a,b,c},{a,b,f},{b,c,f},{b,d,f},{b,e,f},{c,e,f},{a,b,c,f},{a,b,d,f},{a,b,e,f},{b,c,d,f},{b,c,e,f},{b,d,e,f},{a,b,c,d,f},{a,b,c,e,f},{a,b,d,e,f},{b, c, d, e, f,}, X}.

Here $\tau \subsetneq S^*O(X, \tau) \subsetneq SO(X, \tau)$.

Example 3.35: Consider the subspace (X, τ) of the digital plane where X= {0, 1, 2} × {1, 2}.

If a, b, c, d, e, f denote the points (0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2) respectively, then

τ={ϕ,{c},{a,c},{c,d},{c,e},{a,c,d},{a,c,e},{c,d,e},{a,b,c,d},{a,c,d,e},{c,d,e,f},{a,b,c,d,e},{a, c, d, e, f}, X}.

SO(X)= S*O(X)={ϕ,{c},{a,c},{b,c},{c,d},{c,e},{c,f},{a,b,c},{a,c,d},{a,c,e},{a,c,f},{b,c,d}, {b, c, e},{b, c, f},{c,d,e},{c,d,f},{c,e,f},{a,b,c,d},{a,b,c,e},{a,c,d,e},{a,b,c,f},{a,c,d,e},{a,c,d,f}, {a, c, e, f}, {b, c, d, e}, {b, c, d, f},{b,c,e,f},{c,d,e,f},{a,b,c,d,e},{a,b,c,d,f},{a,c,d,e,f},{a,b,c,e,f}, {b, c, d, e, f}, X}. Here $\tau \subsetneq S^*O(X, \tau) = SO(X, \tau)$.

Remark 3.36: If X is a $T_{1/2}$ space, the g-closed sets and the closed sets coincide and hence $Cl^*(U) = Cl(U)$. Therefore the class of semi*-open sets and the class of semi-open sets coincide. In particular, in the Khalimsky line and in the real line with usual topology, the semi*-open sets and the semi-open sets coincide. But the converse is not true. That is, a space, in which the class of semi*-open sets and the class of semi-open sets coincide, need not be $T_{1/2}$ and this can be seen from the Example 3.31 and Example 3.35. In these spaces the class of semi*-open sets and the class of semi-open sets coincide but they are not $T_{1/2}$.

Theorem 3.37: If (X, τ) is any topological space, then $S^*O(X, \tau^*) = SO(X, \tau^*)$.

Proof: Follows from the fact that the space (X, τ^*) is $T_{1/2}$ [Theorem 2.10] and Remark 3.36.

Lemma 3.38: If A be semi*-open, then $Cl^*(A) = Cl^*(Int(A))$.

Proof: Since A is semi*-open, $A \subseteq Cl^*(Int(A))$. Hence $Cl^*(A) \subseteq Cl^*(Int(A))$ which proves the lemma.

Theorem 3.39: Let A be semi*-open and $B \subseteq X$ such that $A \subseteq B \subseteq Cl^*(A)$.Then B is semi*-open.

Proof: Since A is semi*-open, $A \subseteq Cl^*(Int(A))$. Since $Int(A) \subseteq Int(B)$, $Cl^*(Int(A)) \subseteq Cl^*(Int(B))$. Therefore by the above lemma, $B \subseteq Cl^*(Int(B))$. Hence by Theorem 3.4, B is semi*-open.

Theorem 3.40: Let ß be a collection of subsets in (X, τ) satisfying (i) $\tau \subseteq$ ß (ii) If B∈ß and $D \subseteq X$ such that $B \subseteq D \subseteq Cl^*(B)$ implies D∈ß. Then $S^*O(X, \tau) \subseteq$ ß. Thus $S^*O(X, \tau)$ is the smallest collection satisfying the conditions (i) and (ii).

Proof: Let $A \in S^*O(X, \tau)$.Then there is an open set U in X such that $U \subseteq A \subseteq Cl^*(U)$. By (i), U∈ß. By (ii), A∈ß. Thus $S^*O(X, \tau) \subseteq$ ß. Also by Theorem 3.17 and Theorem 3.39, $S^*O(X, \tau)$ satisfies (i) and (ii).Thus $S^*O(X, \tau)$ is the smallest collection satisfying (i) and (ii).

Theorem 3.41: If (X, τ) is a topological space, then $S^*O(X, \tau) \subseteq SO(X, \tau^*)$

That is, every semi*-open set in (X, τ) is semi-open in (X, τ^*).

Proof: If A is a semi*-open set in (X, τ), then there is an open set U in (X, τ) such that $U \subseteq A \subseteq Cl^*(U)$.Since U is open in (X, τ), U is open in (X, τ^*). Thus A is semi-open in (X, τ^*).

Remark 3.42: The inclusion in Theorem 3.41 can be strict and equality also holds as seen from the following examples:

Example 3.43: Consider the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a, b\}, \{a, b, c\}, X\}$

$S^*O(X, \tau) = \{\phi, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. $GC(X, \tau) = \{\phi, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

$\tau^*=\{\phi,\{a\},\{b\},\{c\},\{a, b\},\{a, c\},\{b, c\},\{a, b, c\},X\}$.

$SO(X, \tau^*) =\{\phi,\{a\},\{b\},\{c\},\{a, b\},\{a, c\},\{a, d\},\{b, c\},\{b, d\},\{c, d\},\{a, b, c\}, \{a, b, d\},\{a, c, d\}, \{b, c, d\}, X\}$
$= \wp(X) \setminus \{\{d\}\}$.

Here $S^*O(X, \tau) \subsetneq SO(X, \tau^*)$.

Example 3.44: Consider the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi,\{a\},\{b\},\{c\},\{a, b\},\{a, c\}, \{b, c\},\{a, b, c\},X\}$. $S^*O(X, \tau) =\{\phi,\{a\},\{b\},\{c\},\{a, b\},\{a, c\},\{a, d\},\{b, c\},\{b, d\},\{c, d\},\{a, b, c\}, \{a, b, d\},\{a, c, d\}, \{b, c, d\}, X\}$; $GC(X, \tau)= \{\phi,\{d\},\{a, d\},\{b, d\},\{c, d\},\{a, b, d\},\{a, c, d\}, \{b, c, d\}, X\}$.

$\tau^*=\{\phi,\{a\},\{b\},\{c\},\{a, b\},\{a, c\},\{b, c\},\{a, b, c\}, X\}$.

Here $SO(X, \tau^*) = S^*O(X, \tau) = \wp(X)\setminus\{\{d\}\}$.

Remark 3.45: The concepts of semi*-open sets and α-open sets are independent as seen from the following examples:

Example 3.46: In the space (X, τ) where $X= \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b, c\}, X\}$, the subsets $\{a, b\}$, $\{a, c\}$, $\{a, b, d\}$ and $\{a, c, d\}$ are α-open but not semi*-open.

Example 3.47: In the space (X, τ) where $X= \{a, b, c, d\}$ and $\tau= \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$, the subsets $\{a, d\}$ and $\{b, d\}$ are semi*-open but not α-open.

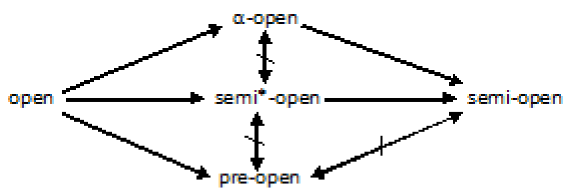
Remark 3.48: The concepts of semi*-open sets and pre-open sets are independent as seen from the following examples:

Example 3.49: In the space (X, τ) where $X= \{a, b, c, d\}$ and $\tau= \{\phi, \{a\}, X\}$, the subsets $\{a, b\}$,$\{a, c\}$,$\{a, d\}$,$\{a, b, c\}$,$\{a, b, d\}$ and $\{a, c, d\}$ are pre-open but not semi*-open.

Example 3.50: In the space (X, τ) where $X= \{a, b, c, d\}$ and $\tau= \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$, the subsets $\{a, d\}$, $\{b, d\}$ and $\{b, c, d\}$are semi*-open but not pre-open.

From the above discussions we have the following diagram:

Diagram 3.51:



Theorem 3.52: In any topological space (X, τ) the following hold:
(i) $s^*Int(\phi)=\phi$.
(ii) $s^*Int(X)=X$.

If A and B are subsets of X,

(iii) $s^*Int(A)\subseteq A$.
(iv) $A\subseteq B \Longrightarrow s^*Int(A)\subseteq s^*Int(B)$.
(v) $s^*Int(s^*Int(A))=s^*Int(A)$. That is, the operator s^*Int is idempotent.
(vi) $Int(A)\subseteq s^*Int(A)\subseteq sInt(A)\subseteq A$.
(vii) $s^*Int(A\cup B)\supseteq s^*Int(A)\cup s^*Int(B)$.
(viii) $s^*Int(A\cap B)\subseteq s^*Int(A)\cap s^*Int(B)$.
(ix) $Int(s^*Int(A))=Int(A)$.
(x) $s^*Int(Int(A))=Int(A)$.

Proof: (i), (ii), (iii) and (iv) follow from Definition 3.2. (v) follows from Theorem 3.13 and Theorem 3.14. (vi) follows from Theorem 3.17 and Theorem 3.24. (vii) and (viii) follow from (iv) above. Since $s^*Int(A)\subseteq A$, $Int(s^*Int(A))\subseteq Int(A)$.

Also from (vi), $Int(A)\subseteq s^*Int(A)$ and so $Int(A)\subseteq Int(s^*Int(A))$.Therefore $Int(s^*Int(A))=Int(A)$. This proves (ix). (x) follows from the fact that $Int(A)$ is open and hence semi*-open and by invoking Theorem 3.14, $s^*Int(Int(A))=Int(A)$.

Remark 3.53: In (vi) of Theorem 3.52, each of the inclusions may be strict and equality may also hold. This can be seen from the following examples:

Example 3.54: In the space (X, τ) where X={a, b, c, d, e, f, g} and τ={ϕ,{a},{b},{a, b}, {a, b, e},{a, c, d},{b, f, g}, {a, b, c, d},{a, b, f, g},{a, b, c, d, e},{a, b, e, f, g},{a, b, c, d, f, g}, X}. Let A= {a, b, c, d}.

Then $Int(A)=s^*Int(A)=sInt(A)$={a, b, c, d}=A.

Let B={a, e}.Then $Int(B)$={a}; $s^*Int(B)=sInt(B)$={a, e}.

Here $Int(B)\subsetneq s^*Int(B)=sInt(B)=B$.

Let C= {a, b, c, d, e, f}.Then $Int(C)=s^*Int(C)$={a, b, c, d, e}; $sInt(C)$={a, b, c, d, e, f}.

Here $Int(C) =s^*Int(C) \subsetneq sInt(C) =C$.

Let D= {b, d, f, g}.Then $Int(D)=s^*Int(D)=sInt(D)$={b, f, g}. Here $Int(D)=s^*Int(D)=sInt(D)\subsetneq D$.

Let E={a, c, e}.Then $Int(E)$={a}; $s^*Int(E)$={a, e}; $sInt(E)$={a, c, e}.

Here $Int(E)\subsetneq s^*Int(E)\subsetneq sInt(E)=E$.

Let F= {b, c, d, e}.Then $Int(F)$={b}; $s^*Int(F)=sInt(F)$={b, e}.

Here $Int(F)\subsetneq s^*Int(F)= sInt(F)\subsetneq F$.

Let G={a, d, f}.Then $Int(G)=s^*Int(G)$={a}; $sInt(G)$={a, d}.

Here $Int(G)=s^*Int(G)\subsetneq sInt(G)\subsetneq G$. Let H= {b, c, d, e, f}.

Then $Int(H)$={b}; $s^*Int(H)$={b, e}; $s^*Int(H)$={b, e, f}.

Here $Int(H) \subsetneq s^*Int(H)\subsetneq sInt(H) \subsetneq H$.

Example 3.55: Consider the subspace (X, τ) of the digital plane where X= {1, 2, 3} × {1, 2}.

If a, b, c, d, e, f denote the points (1,1),(1,2),(2,1),(2,2),(3,1),(3,2) respectively, then

τ={ϕ,{a},{e},{a,b},{a,e},{e,f},{a,b,e},{a,c,e},{a,e,f},{a,b,c,e},{a,b,e,f},{a,c,e,f},{a,b,c,e,f},X}.

SO(X)={ϕ,{a},{e},{a,b},{a,c},{a,d},{a,e},{c,e},{d,e},{e,f},{a,b,c}{a,b,d},{a,b,e},{a,c,d},{a,c,e},{a,d,e},{a,e,f},{c,d, e},{c,e,f},{d,e,f},{a,b,c,d},{a,b,c,e},{a,b,d,e}{a,b,e,f},{a,c,d,e},{a,c,e,f},{a,d,e,f},{c,d,e,f},{a,b,c,d,e},{a,b,c,e,f},{a,b, d,e,f},{a,c,d,e,f},X}.

S*O(X)={ϕ,{a},{e},{a,b},{a,d},{a,e},{d,e},{e,f},{a,b,d},{a,b,e},{a,c,e},{a,d,e},{a,e,f},{d,e,f},{a,b,c,e},{a,b,d,e}{a,b, e,f},{a,c,d,e},{a,c,e,f},{a,d,e,f},{a,b,c,d,e},{a,b,c,e,f},{a,b,d,e,f},{a,c,d,e,f}, X}.

Let A={a, b, c, d, f}.Then $Int(A)$={a, b}; $s^*Int(A)$={a, b, d}; $sInt(A)$={a, b, c, d}.

Here $Int(A)\subsetneq s^*Int(A)\subsetneq sInt(A)\subsetneq A$.

Let B={b, c, e, f}.Then $Int(B)=s^*Int(B)$={e, f}; $sInt(B)$={c, e, f}.

Here $Int(B)=s^*Int(B)\subsetneq sInt(B)\subsetneq B$.

Let C= {a, b, d, f}.Then $Int(C)$={a, b}; $s^*Int(C)= sInt(C)$={a, b, d}. Here $Int(C)\subsetneq s^*Int(C)=sInt(C)\subsetneq C$.

Let D= {a, c, d}. Then Int(D)={a}; $s*Int$(D)={a, d};$sInt$(D)={a, c, d}.

Here $Int(D) \subsetneq s*Int(D) \subsetneq sInt(D)=D$.

Let E={a, b, c, d, e}.Then Int(E)={a, b, c, e}; $s*Int$(E)= $sInt$(E)={a, b, c, d, e}.

Here $Int(E) \subsetneq s*Int(E)=sInt(E)=E$.

Let F= {a, c}. Then Int(F)= $s*Int$(F)={a} ; $sInt$(F)={a, c}. Here $Int(F) = s*Int(F) \subsetneq sInt(F) = F$.

Let G= {b, e, f}. Then Int(G)= $s*Int$(G)= $sInt$(G)={e,f}. Here $Int(G) = s*Int(G)= sInt(G) \subsetneq G$.

Let H= {a, b, e, f}. Then $Int(H)=s*Int(H)= sInt(H) = H$.

Remark 3.56: The inclusions in (vii) and (viii) of Theorem 3.52 may be strict and equality may also hold. This can be seen from the following examples.

Example 3.57: Consider the space (X, τ) in Example 3.54

Let A= {b, c, e, f, g} and B={a, b, c, f, g} then A∪B={a, b, c, e, f, g} and A∩B={b, c, f, g}
$s*Int$(A)={b, e, f, g}; $s*Int$(B)={a, b, f, g}; $s*Int$(A∪B)={a, b, e, f, g}; $s*Int$(A∩B)={b, f, g}

Here $s*Int(A\cup B)= s*Int(A) \cup s*Int(B)$ and $s*Int(A\cap B)= s*Int(A) \cap s*Int(B)$

Let C= {a, c, d, e, g} and D={b, d, e, f, g} then C∩D={d, e, g}

$s*Int$(C)={a, c, d, e}; $s*Int$(D)={ b, e, f, g}; $s*Int(C\cap D)=\phi$; $s*Int(C) \cap s*Int(D)$={e}

Here $s*Int(C\cap D) \subsetneq s*Int(C) \cap s*Int(D)$

Let E={b, c, d, f, g} and F={a, b, d, g} then E∪F={a, b, c, d, f, g};$s*Int$(E)={b, f, g};

$s*Int$(F)={a, b}; $s*Int$(E∪F)={a, b, c, d, f, g}; $s*Int(E) \cup s*Int(F)$={a, b, f, g};

Here $s*Int(E) \cup s*Int(F) \subsetneq s*Int(E\cup F)$

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