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SOME RESULTS BASED ON RELATIVE DEFECTS OF SPECIAL TYPE OF DIFFERENTIAL POLYNOMIALS

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ABSTRACT

The aim of this paper is to compare the relative Valiron defect with the relative Nevanlinna defect of special type of differential polynomials generated by transcendental meromorphic functions.

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1. INTRODUCTION, DEFINITIONS AND NOTATIONS

Let f be a transcendental meromorphic function defined in the open complex plane C. A monomial in f is an expression of the form $M[f] = (f)^{n_0}\, (f^{(1)})^{n_1} \ldots (f^{(k)})^{n_k}$ where $n_0, n_1, n_2, \ldots, n_k$ are non negative integers. $\gamma_M = n_0 + n_1 + \cdots + n_k$ and $\Gamma_M = n_0 + 2n_1 + \ldots + (k+1)n_k$ are respectively called the degree and weight of the monomial.

If $M_1[f], M_2[f], \ldots, M_n[f]$ denote monomials in f, then $Q[f] = a_1\, M_1[f] + a_2\, M_2[f] + \cdots + a_n\, M_n[f]$, where $a_i \neq 0$(i=1,2,…,n) is called a differential polynomial generated by f of degree $\gamma_Q = Max\,\{\gamma_{M_j} : 1 \leq j \leq n\}$ and weight $\Gamma_Q = Max\{\Gamma_{M_j} : 1 \leq j \leq n\}$.

Also we call numbers $\underline{\gamma_Q} = \min_{1 \leq j \leq n} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $Q[f]$ respectively. If $\underline{\gamma_Q} = \gamma_Q$, $Q[f]$ is called a homogeneous differential polynomial.

For $a \in C \cup \{\infty\}$, the quantity

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

is called the Nevanlinna's deficiency of the value 'a'. Similarly the Valiron defect of 'a' is defined as

$$\Delta(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \limsup_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}.$$

The term $\delta_R^{(k)}(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f^{(k)})}{T(r,f)}$ for $k = 1,2,3,\ldots$ is called the relative Nevanlinna's defect of 'a' with respect to $f^{(k)}$. In a like manner $\Delta_R^{(k)}(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;f^{(k)})}{T(r,f)}$ for $k = 1,2,3,\ldots$ is called the relative Valiron defect of 'a' with respect to $f^{(k)}$. Xiong [3] has shown various relations between the usual defects and relative defects of meromorphic functions. Following Datta and Mondal [1], in the paper we consider $F = f^n Q[f]$, $Q[f]$ being a differential polynomial in f and $n = 1,2,3,\ldots$ and compare the relative Valiron defect with the relative Nevanlinna defect of F.

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The term $S(r,f)$denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ through all values of r if $f$is of finite order and except possibly for a set of r of finite linear measure otherwise. We do not explain the standard definitions and notations of the value distribution and the Nevanlinna theory as those are available in [2].

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 1: Let k be any positive integer and $\varphi = \sum_{i=0}^{n} a_i f^{(i)}$, where a_i are meromorphic functions such that $T(r,a_i) = S(r,f)$ for$i = 0,1,2,\dots,k$.

Lemma 2: Let $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f. If $n \geq 1$ then $\lim_{r\to\infty} \frac{T(r,F)}{T(r,f)} = 1$.

The proof is omitted.

Lemma 3: Let $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f. If $n \geq 1$ then for any

α, $\delta_R^F(\alpha;f) = \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,f)}$ and $\Delta_R^F(\alpha;f) = \limsup_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,f)}$.

Proof: In view of Lemma 2 we get that

$$\begin{aligned}
\delta_R^F(\alpha;f) &= 1 - \limsup_{r\to\infty} \frac{N(r,\alpha;F)}{T(r,f)} \\
&= 1 - \limsup_{r\to\infty} \frac{N(r,\alpha;F)}{T(r,F)}.\ \lim_{r\to\infty} \frac{T(r,F)}{T(r,f)} \\
&= 1 - \limsup_{r\to\infty} \frac{N(r,\alpha;F)}{T(r,F)}.1 \\
&= 1 - \limsup_{r\to\infty} \frac{N(r,\alpha;F)}{T(r,F)} \\
&= \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,F)} \\
&= \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,f)}.\ \lim_{r\to\infty} \frac{T(r,f)}{T(r,F)} \\
&= \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,f)}.1 \\
&= \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,f)}.
\end{aligned}$$

This proves the first part of the lemma.

Similarly the second part of Lemma 3 follows.

3. THEOREMS.

In this section we present the main results of the paper.

Theorem 1: Let f be a transcendental meromorphic function of finite order ρ_f and satisfying the condition $m(r,f) = S(r,f)$. If a, b and c are three non zero finite complex numbers then

$$3\delta(a;f) + 2\delta(b;f) + \delta(c;f) + 5\Delta_R^F(\infty;F) \leq 5\Delta(\infty;f) + 5\Delta_R^F(\infty;F)$$

where F is a differential polynomial in f of the form $F = f^n Q[f]$ with $n \geq 1$.

Proof: Let us consider the following identity

$$\frac{b-a}{f-a} = \left[\frac{F}{f-a}\left\{\frac{f-a}{F} - \frac{f-b}{F}\right\} - \frac{f-c}{F}.\frac{F}{f}.\frac{F}{f-a}\left\{\frac{f-a}{F} - \frac{f-b}{F}\right\}\right].\frac{f}{c}$$

Since $m\left(r,\frac{1}{f-a}\right) \leq m\left(r,\frac{b-a}{f-a}\right) + O(1)$ and $m\left(r,\frac{f}{c}\right) \leq m(r,f) + O(1)$, we get from the above identity in view of Lemma 1 that

$$m\left(r,\frac{b-a}{f-a}\right) \leq m\left(r,\frac{f-a}{F}\right) + m\left(r,\frac{f-b}{F}\right) + m\left(r,\frac{f-a}{F}\right) + m\left(r,\frac{f-b}{F}\right) + m\left(r,\frac{f-c}{F}\right) + m\left(r,\frac{f}{c}\right) + S(r,f)$$

$$i.e., m\left(r,\frac{1}{f-a}\right) \le 2m\left(r,\frac{f-a}{F}\right) + 2m\left(r,\frac{f-b}{F}\right) + m\left(r,\frac{f-c}{F}\right) + m(r,f) + S(r,f) + O(1)$$

$$i.e., m\left(r,\frac{1}{f-a}\right) \le 2T\left(r,\frac{f-a}{F}\right) - 2N\left(r,\frac{f-a}{F}\right) + 2T\left(r,\frac{f-b}{F}\right) - 2N\left(r,\frac{f-b}{F}\right) + T\left(r,\frac{f-c}{F}\right) - N\left(r,\frac{f-c}{F}\right) + m(r,f) + S(r,f) + O(1). \quad (1)$$

Now by the relation $T\left(r,\frac{1}{f}\right) = T(r,f) + O(1)$ and in view of Lemma 1 it follows from (1) that

$$m\left(r,\frac{1}{f-a}\right) \le 2T\left(r,\frac{F}{f-a}\right) - 2N\left(r,\frac{f-a}{F}\right) + 2T\left(r,\frac{F}{f-b}\right) - 2N\left(r,\frac{f-b}{F}\right) + T\left(r,\frac{F}{f-c}\right) - N\left(r,\frac{f-c}{F}\right) + m(r,f) + S(r,f) + O(1)$$

$$i.e., m\left(r,\frac{1}{f-a}\right) \le 2\left\{N\left(r,\frac{F}{f-a}\right) - N\left(r,\frac{f-a}{F}\right)\right\} + 2\left\{N\left(r,\frac{F}{f-b}\right) - N\left(r,\frac{f-b}{F}\right)\right\} + N\left(r,\frac{F}{f-c}\right) - N\left(r,\frac{f-c}{F}\right) + m(r,f) + S(r,f) + O(1). \quad (2)$$

In view of {p.34, [2]} it follows from (2) that

$$m\left(r,\frac{1}{f-a}\right) \le 2\left\{N(r,F) + N\left(r,\frac{1}{f-a}\right) - N(r,f-a) - N\left(r,\frac{1}{F}\right)\right\} + 2\left\{N(r,F) + N\left(r,\frac{1}{f-b}\right) - N(r,f-b) - N\left(r,\frac{1}{F}\right)\right\} + \left\{N(r,F) + N\left(r,\frac{1}{f-c}\right) - N(r,f-c) - N\left(r,\frac{1}{F}\right)\right\} + m(r,f) + S(r,f) + O(1). \quad (3)$$

Now applying the condition $m(r,f) = S(r,f)$ it follows from (3) that

$$m\left(r,\frac{1}{f-a}\right) \le 5N(r,F) - 5N(r,f) - 5N\left(r,\frac{1}{F}\right) + 2N\left(r,\frac{1}{f-a}\right) + 2N\left(r,\frac{1}{f-b}\right) + N\left(r,\frac{1}{f-c}\right) + S(r,f)$$

$$i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)} \le 5\liminf_{r\to\infty}\left\{\frac{N(r,F)}{T(r,f)} - \frac{N(r,f)}{T(r,f)} - \frac{N\left(r,\frac{1}{f}\right)}{T(r,f)}\right\} + 2\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)} + 2\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-b}\right)}{T(r,f)} + \limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-c}\right)}{T(r,f)}$$

$$i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)} \le 5\liminf_{r\to\infty}\frac{N(r,F)}{T(r,f)} - 5\liminf_{r\to\infty}\frac{N(r,f)}{T(r,f)} - 5\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{f}\right)}{T(r,f)} + 2\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)} + 2\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-b}\right)}{T(r,f)} + \limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-c}\right)}{T(r,f)}$$

$$i.e., \delta(a;f) \le 5\{1 - \Delta_R^F(\infty;f)\} - 5\{1 - \Delta(\infty;f)\} - 5\{1 - \Delta_R^F(0;f)\} + 2\{1 - \delta(a;f)\} + 2\{1 - \delta(b;f)\} + \{1 - \delta(c;f)\}$$

$$i.e., 3\delta(a;f) + 2\delta(b;f) + \delta(c;f) + 5\Delta_R^F(\infty;f) \le 5\Delta(\infty;f) + 5\Delta_R^F(0;f).$$

This proves the theorem.

Theorem 2: Let f be a meromorphic function of finite order satisfying the condition $m(r,f) = S(r,f)$ and also let If $F = f^n Q[f]$ is a differential polynomial in f with $n \ge 1$. If a, b, c and d are any four distinct complex numbers then

$$\delta(d;f) + \delta_R^F(b;f) + \delta_R^F(c;f) \le 2.$$

Proof: Let us consider the following identity

$$\frac{1}{f-d} = \left[\frac{1}{a}\left\{\frac{F}{f-a} - \frac{F-a}{f^n}.\frac{f^n}{f-a}\right\}.\left\{\frac{F}{f-d}.\frac{1}{F}\right\}\right].(f-a)$$

Since $m(r,f-a) \le m(r,f) + O(1)$, we get from the above identity in view of Lemma 1 that

$$m\left(r,\frac{1}{f-d}\right) \le m\left(r,\frac{1}{F}\right) + m(r,f) + S(r,f) + O(1)$$

$$i.e., m\left(r,\frac{1}{f-d}\right) \le T\left(r,\frac{1}{F}\right) - N\left(r,\frac{1}{F}\right) + m(r,f) + S(r,f) + O(1)$$

Now by the relation $T\left(r,\frac{1}{F}\right) = T(r,f) + O(1)$ we get from the above

$$m\left(r,\frac{1}{f-d}\right) \le T(r,F) - N\left(r,\frac{1}{F}\right) + m(r,f) + S(r,f) + O(1). \quad (4)$$

Now by Nevanlinna's second fundamental theorem it follows from (4) that

$$m\left(r,\frac{1}{f-d}\right) \le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-b}\right) + \overline{N}\left(r,\frac{1}{F-c}\right) - N\left(r,\frac{1}{F}\right) + m(r,f) + S(r,f) + O(1). \quad (5)$$

As $\overline{N}\left(r,\frac{1}{F}\right) - N\left(r,\frac{1}{F}\right) \le 0$ and applying the condition $m(r,f) = S(r,f)$ it follows from (5) that

$$m\left(r,\frac{1}{f-d}\right) \le \overline{N}\left(r,\frac{1}{F-b}\right) + \overline{N}\left(r,\frac{1}{F-c}\right) + S(r,f)$$
$$i.e., m\left(r,\frac{1}{f-d}\right) \le N\left(r,\frac{1}{F-b}\right) + N\left(r,\frac{1}{F-c}\right) + S(r,f)$$

$$i.e., m\left(r,\frac{1}{f-d}\right) \le T\left(r,\frac{1}{F-b}\right) - m\left(r,\frac{1}{F-b}\right) + T\left(r,\frac{1}{F-c}\right) - m\left(r,\frac{1}{F-c}\right) + S(r,f)$$

$$i.e., m\left(r,\frac{1}{f-d}\right) \le T(r,F) - m\left(r,\frac{1}{F-b}\right) - m\left(r,\frac{1}{F-c}\right) + S(r,f)$$

$$i.e., \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-d}\right)}{T(r,f)} \le 2\liminf_{r\to\infty}\frac{T(r,F)}{T(r,f)} - \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-b}\right)}{T(r,f)} - \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-c}\right)}{T(r,f)}$$

$$i.e., \delta(d;f) \le 2.1 - \delta_R^F(b;f) - \delta_R^F(c;f)$$

$$i.e., \delta(d;f) + \delta_R^F(b;f) + \delta_R^F(c;f) \le 2.$$

This proves the theorem.

Theorem 3: Let f be a transcendental meromorphic function of finite order ρ_f and satisfying the condition $m(r,f) = S(r,f)$. If a and c are any two distinct complex numbers and let $F = f^nQ[f]$ is a differential polynomial in f with $n \ge 1$ then

$$\delta(0;f) + \delta(c;f) + \Delta_R^F(\infty;f) \le \Delta(\infty;f) + 2\Delta_R^F(0;f).$$

Proof: Consider the following identity

$$\frac{c}{f} = \left[\left\{1 - \frac{f-c}{F}.\frac{F}{f}\right\}\left\{\frac{F}{f-a}.\frac{1}{F}\right\}\right].(f-a)$$

Since $m\left(r,\frac{1}{f}\right) \le m\left(r,\frac{c}{f}\right) + O(1) and\ m(r,f-a) \le m(r,f) + O(1)$, we get from the above identity in view of Lemma 1 that

$$m\left(r,\frac{c}{f}\right) \le m(r,\frac{f-c}{F}) + m\left(r,\frac{1}{F}\right) + m(r,f) + S(r,f) + O(1)$$

$$i.e., m\left(r,\frac{1}{f}\right) \le T\left(r,\frac{f-c}{F}\right) - N\left(r,\frac{f-c}{F}\right) + T\left(r,\frac{1}{F}\right) - N\left(r,\frac{1}{F}\right) + m(r,f) + S(r,f) + O(1). \quad (6)$$

Now by Nevanlinna's first fundamental theorem and in view of Lemma 1 it follows from (6) that

$$m\left(r,\frac{1}{f}\right) \le T\left(r,\frac{F}{f-c}\right) - N\left(r,\frac{f-c}{F}\right) + T(r,F) - N\left(r,\frac{1}{F}\right) + m(r,f) + S(r,f) + O(1).$$

$$i.e., m\left(r,\frac{1}{f}\right) \leq N\left(r,\frac{F}{f-c}\right) - N\left(r,\frac{f-c}{F}\right) - N\left(r,\frac{1}{F}\right) + T(r,F) + m(r,f) + S(r,f) + O(1). \quad (7)$$

In view of {p.34, [2]} it follows from (7) that

$$m\left(r,\frac{1}{f}\right) \leq N(r,F) + N\left(r,\frac{1}{f-c}\right) - N(r,f-c) - N\left(r,\frac{1}{F}\right) - N\left(r,\frac{1}{F}\right) + T(r,F) + m(r,f) + S(r,f) + O(1)$$

$$i.e., \liminf_{r\to\infty}\frac{m(r,\frac{1}{f})}{T(r,f)} \leq \liminf_{r\to\infty}\frac{N(r,F)}{T(r,f)} - \liminf_{r\to\infty}\frac{N(r,f)}{T(r,f)} - 2\liminf_{r\to\infty}\frac{N(r,\frac{1}{F})}{T(r,f)} + \limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-c}\right)}{T(r,f)} + \limsup_{r\to\infty}\frac{T(r,F)}{T(rf)}$$

$$i.e., \delta(0;f) \leq \{1-\Delta_R^F(\infty;f)\} - \{1-\Delta(\infty;f)\} - 2\{1-\Delta_R^F(0;f)\} + \{1-\delta(c;f)\} + 1$$

$$i.e., \delta(0;f) + \delta(c;f) + \Delta_R^F(\infty;f) \leq \Delta(\infty;f) + 2\Delta_R^F(0;f).$$

Thus the theorem is established.

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