

COEFFICIENT INEQUALITIES FOR GENERALIZED SAKAGUCHI TYPE
MULTIVALENT FUNCTIONS

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ABSTRACT

In this paper we have studied two subclasses $S_p(\alpha, s, t)$ and $T_p(\alpha, s, t)$ concerning with generalized Sakaguchi type functions in the open unit disc U , further by using the coefficient inequalities for the classes $S_p(\alpha, s, t)$ and $T_p(\alpha, s, t)$, two subclasses $S_p^0(\alpha, s, t)$ and $T_p^0(\alpha, s, t)$ are defined. Some properties of functions belonging to the class $S_p^0(\alpha, s, t)$ and $T_p^0(\alpha, s, t)$ are also discussed.

Keywords: Analytic functions; Starlike functions; Convex functions; Sakaguchi function; Coefficient inequalities.

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1. INTRODUCTION

Let A_p be the class of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \tag{1.1}$$

that are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in A_p$ is said to be in the class $S_p(\alpha, s, t)$ defined by Frasin [1] if it satisfies

$$\operatorname{Re} \left\{ \frac{(s-t)zf'(z)}{f(sz)-f(tz)} \right\} > \alpha \tag{1.2}$$

for some $\alpha (0 \leq \alpha < 1)$, $s, t \in \mathbb{C}, s \neq t$ and for all $z \in U$. For $p = 1$ and $s = 1$, the class $S_1(\alpha, 1, t) = S(\alpha, 1, t)$ was introduced and studied by Owa et al.[4], and by taking $t = -1$, the class $S_1(\alpha, 1, -1) = S_s(\alpha)$ was introduced by Sakaguchi [3] and is called Sakaguchi function of order α (see [1],[5]), where $S_s(0) = S_s$ is the class of starlike functions with respect to symmetrical points in U . We also denote by $T_p(\alpha, s, t)$ the subclass of A_p consisting of all functions $f(z)$ such that $zf'(z) \in S_p(\alpha, s, t)$. Also, we note that $S_1(\alpha, 1, 0) = S^*(\alpha)$ and $T_1(\alpha, 1, 0) = T^*(\alpha)$ which are, respectively, the familiar classes of starlike functions of order $\alpha (0 \leq \alpha < 1)$ and convex functions of order $\alpha (0 \leq \alpha < 1)$.

2. COEFFICIENT INEQUALITIES FOR SUBCLASSES $S_p^0(\alpha, s, t)$ AND $T_p^0(\alpha, s, t)$

We first prove the following two theorems which are similar to the result of Cho et al.[3] and Owa et al.[4]

Theorem 2.1: If $f(z) \in A_p$ satisfies

$$\sum_{n=1}^{\infty} [|p+n-u_{p+n}| + (1-\alpha)|u_{p+n}|] |a_{p+n}| \leq p-\alpha|u_p| \tag{2.1}$$

for some $\alpha (0 \leq \alpha < 1)$, then $f(z) \in S_p(\alpha, s, t)$ where

$$u_p = \sum_{j=1}^p s^{p-j} t^{j-1} \tag{2.2}$$

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Proof: To prove Theorem 2.1, we show that if $f(z)$ satisfies (2.1) then

$$\left| \frac{(s-t)zf'(z)}{f(sz) - f(tz)} - 1 \right| < 1 - \alpha$$

Evidently, since

$$\begin{aligned} \frac{(s-t)zf'(z)}{f(sz) - f(tz)} - 1 &= \frac{(s-t)\{pz^p + \sum_{n=1}^{\infty} (p+n)a_{p+n}z^{p+n}\}}{\{(sz)^p + \sum_{n=1}^{\infty} a_{p+n}(sz)^{p+n}\} - \{(tz)^p + \sum_{n=1}^{\infty} a_{p+n}(tz)^{p+n}\}} - 1 \\ &= \frac{(p-u_p)z^p + \sum_{n=1}^{\infty} (p+n-u_{p+n})a_{p+n}z^{p+n}}{u_pz^p + \sum_{n=1}^{\infty} u_{p+n}a_{p+n}z^{p+n}} \end{aligned}$$

We see that

$$\left| \frac{(s-t)zf'(z)}{f(sz) - f(tz)} - 1 \right| \leq \frac{|p-u_p| + \sum_{n=1}^{\infty} |p+n-u_{p+n}||a_{p+n}|}{|u_p| + \sum_{n=1}^{\infty} |u_{p+n}||a_{p+n}|}$$

Therefore, if $f(z)$ satisfies (2.1), then we have

$$\frac{|p-u_p| + \sum_{n=1}^{\infty} |p+n-u_{p+n}||a_{p+n}|z^{p+n}}{|u_p| + \sum_{n=1}^{\infty} |u_{p+n}||a_{p+n}|} \leq 1 - \alpha$$

This completes the proof of Theorem 2.1.

Theorem 2.2: If $f(z) \in A_p$ satisfies

$$\sum_{n=1}^{\infty} (p+n)[|p+n-u_{p+n}| + (1-\alpha)|u_{p+n}||a_{p+n}|] \leq p^2 - \alpha p|u_p| \quad (2.3)$$

or some $\alpha(0 \leq \alpha < 1)$, then $f(z) \in T_p(\alpha, s, t)$.

Proof: Noting that $f(z) \in T_p(\alpha, s, t)$ if and only if $zf'(z) \in T_p(\alpha, s, t)$, we can prove Theorem 2.2.

We now define

$$S_p^0(\alpha, s, t) = \{f(z) \in A_p \text{ such that } f(z) \text{ satisfies (2.1)}\}$$

$$T_p^0(\alpha, s, t) = \{f(z) \in A_p \text{ such that } f(z) \text{ satisfies (2.3)}\}$$

3. COEFFICIENT INEQUALITIES FOR SUBCLASSES $S_p(\alpha, s, t)$ AND $T_p(\alpha, s, t)$

Next applying Caratheodry function $\varphi(z)$ defined by

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n \quad (3.1)$$

We discuss the coefficient inequalities for functions $f(z)$ in $S_p(\alpha, s, t)$ and $T_p(\alpha, s, t)$.

Theorem 3.1: If $f(z) \in S_p(\alpha, s, t)$, then

$$|a_{p+n}| \leq \frac{\beta|u_p|}{|\vartheta_{p+n}|} \left\{ 1 + \beta \sum_{j=p+1}^{p+n-1} \frac{|u_j|}{|\vartheta_j|} + \beta^2 \sum_{j_2 > j_1}^{p+n-1} \sum_{j_1=p+1}^{p+n-2} \frac{|u_{j_1}u_{j_2}|}{|\vartheta_{j_1}\vartheta_{j_2}|} + \dots + \beta^{n-1} \prod_{j=p+1}^{p+n-1} \frac{|u_j|}{|\vartheta_j|} \right\} \quad (3.2)$$

where

$$\beta = 2(p - \alpha u_p), \quad \vartheta_n = nu_p - pu_n \quad (3.3)$$

for some $\alpha(0 \leq \alpha < 1)$, $s, t \in \mathbb{C}$, $s \neq t$.

Proof: We define the function $\varphi(z)$ by

$$\varphi(z) = \frac{u_p}{(p-\alpha u_p)} \left(\frac{(s-t)zf'(z)}{f(sz) - f(tz)} - \alpha \right) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n \quad (3.4)$$

for $f(z) \in S_p(\alpha, s, t)$. Then $\varphi(z)$ is caratheodry function and satisfies

$$|\varphi_n| \leq 2 \quad (n \geq 1) \tag{3.5}$$

Since

$$(s-t)zf'(z) = \{f(sz) - f(tz)\} \left\{ \alpha + \frac{(p-\alpha u_p)}{u_p} \varphi(z) \right\}$$

we have

$$(s-t)z^p \left[p + \sum_{n=1}^{\infty} (p+n)a_{p+n}z^n \right] = z^p \left[(s^p - t^p) + \sum_{n=1}^{\infty} (s^{p+n} - t^{p+n})a_{p+n}z^n \right] \left[\frac{p}{u_p} + \frac{(p-\alpha u_p)}{u_p} \sum_{n=1}^{\infty} \varphi_n z^n \right]$$

$$u_p \left[p + \sum_{n=1}^{\infty} (p+n)a_{p+n}z^n \right] = \left[u_p + \sum_{n=1}^{\infty} u_{p+n}a_{p+n}z^n \right] \left[p + (p-\alpha u_p) \sum_{n=1}^{\infty} \varphi_n z^n \right]$$

So we get

$$a_{p+n} = \frac{(p-\alpha u_p)}{((p+n)u_p - pu_{p+n})} [u_{p+n-1}a_{p+n-1}\varphi_1 + u_{p+n-2}a_{p+n-2}\varphi_2 + \dots + u_{p+1}a_{p+1}\varphi_{n-1} + u_p\varphi_n] \tag{3.6}$$

From equation (3.6), we easily have that

$$|a_{p+1}| = \left| \frac{(p-\alpha u_p)}{((p+1)u_p - pu_{p+1})} \varphi_1 u_p \right| \leq 2|(p-\alpha u_p)| \left[\frac{|u_p|}{|(p+1)u_p - pu_{p+1}|} \right]$$

$$|a_{p+2}| = \left| \frac{(p-\alpha u_p)}{((p+2)u_p - pu_{p+2})} [\varphi_2 u_p + u_{p+1}a_{p+1}\varphi_1] \right|$$

$$\leq \frac{2(p-\alpha u_p)|u_p|}{|(p+2)u_p - pu_{p+2}|} \left[1 + 2(p-\alpha u_p) \frac{|u_{p+1}|}{|(p+1)u_p - pu_{p+1}|} \right]$$

$$|a_{p+3}| \leq \frac{2(p-\alpha u_p)|u_p|}{|(p+3)u_p - pu_{p+3}|} \left[1 + \frac{2(p-\alpha u_p)|u_{p+1}|}{|(p+1)u_p - pu_{p+1}|} + \frac{2(p-\alpha u_p)|u_{p+2}|}{|(p+2)u_p - pu_{p+2}|} \right]$$

$$+ \frac{2^2(p-\alpha u_p)^2|u_{p+1}||u_{p+2}|}{|((p+1)u_p - pu_{p+1})| |((p+2)u_p - pu_{p+2})|}$$

Thus, using mathematical induction, we obtain the inequality (3.2).

Remark (1): Equality in Theorem (3.1) are attended for the function $f(z)$ given by

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = 1 + \frac{(1-2\alpha)z}{1-z} \tag{3.7}$$

Remark (2): If we put $s = 1, p = 1, \alpha = 0, t = 0$ in Theorem (3.1), then we have well known result

$$f(z) \in S^* \rightarrow |a_n| \leq n \text{ where } S^* \text{ is a usual starlike class}$$

And if we put $s = 1, p = 1, \alpha = 0, t = -1$, then we have the results due to Sakaguchi [2].

$$f(z) \in STS \rightarrow |a_n| \leq 1 \text{ where STS is Sakaguchi function class.}$$

For the function class $T_p(\alpha, s, t)$ similarly we have,

Theorem 3.2: If $f(z) \in T_p(\alpha, s, t)$, then

$$|a_{p+n}| \leq \frac{\beta p |u_p|}{(p+n)|\vartheta_{p+n}} \left\{ 1 + \beta \sum_{j=p+1}^{p+n-1} \frac{|u_j|}{|\vartheta_j|} + \beta^2 \sum_{j_2 > j_1}^{p+n-1} \sum_{j_1=p+1}^{j_2-1} \frac{|u_{j_1} u_{j_2}|}{|\vartheta_{j_1} \vartheta_{j_2}|} + \dots + \beta^{n-2} \prod_{j=p+1}^{p+n-1} \frac{|u_j|}{|\vartheta_j|} \right\} \tag{3.8}$$

4. DISTORTION INEQUALITIES FOR SUBCLASSES $S_p^0(\alpha, s, t)$ AND $T_p^0(\alpha, s, t)$

For functions $f(z)$ in the classes $S_p^0(\alpha, s, t)$ and $T_p^0(\alpha, s, t)$ we derive

Theorem 4.1: If $f(z) \in S_p^0(\alpha, s, t)$, then

$$|z|^p - \sum_{n=1}^j |a_{p+n}| |z|^{p+n} - A_j |z|^{p+j+1} \leq |z|^p + \sum_{n=1}^j |a_{p+n}| |z|^{p+n} + A_j |z|^{p+j+1} \quad (4.1)$$

Where

$$A_j = \frac{p-\alpha|u_p| - \sum_{n=1}^j [p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}|}{p+j+1-\alpha|u_{p+n}|} \quad (j \geq 1) \quad (4.2)$$

Proof: From the inequality 2.1, we know that

$$\sum_{n=j+1}^{\infty} [p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}| \leq p-\alpha|u_p| - \sum_{n=1}^j [p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}|$$

On the other hand we know that,

$$[p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}| \geq p+n-\alpha|u_{p+n}|$$

and hence $p+n-\alpha|u_{p+n}|$ is monotonically increasing with respect to n . Thus we deduce

$$p+j+1-\alpha|u_{p+n}| \sum_{n=j+1}^{\infty} |a_{p+n}| \leq p-\alpha|u_p| - \sum_{n=1}^j [p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}|$$

which implies that therefore

$$\sum_{n=j+1}^{\infty} |a_{p+n}| \leq A_j$$

Therefore we have that

$$|f(z)| \leq |z|^p + \sum_{n=1}^j |a_{p+n}| |z|^{p+n} + A_j |z|^{p+j+1}$$

And

$$|f(z)| \geq |z|^p - \sum_{n=1}^j |a_{p+n}| |z|^{p+n} - A_j |z|^{p+j+1}$$

This completes the proof.

For function $T_p(\alpha, s, t)$, similarly we have,

Theorem 4.2: If $f(z) \in T_p^0(\alpha, s, t)$, then

$$|z|^p - \sum_{n=1}^j |a_{p+n}| |z|^{p+n} - B_j |z|^{p+j+1} \leq |f(z)| \leq |z|^p + \sum_{n=1}^j |a_{p+n}| |z|^{p+n} + B_j |z|^{p+j+1} \quad (4.4)$$

And

$$p|z|^{p-1} - \sum_{n=1}^j (p+n)|a_{p+n}| |z|^{p+n-1} - C_j |z|^{p+j} \leq |f'(z)| \leq p|z|^{p-1} + \sum_{n=1}^j (p+n)|a_{p+n}| |z|^{p+n-1} + C_j |z|^{p+j} \quad (4.5)$$

where

$$B_j = \frac{p^2-\alpha p|u_p| - \sum_{n=1}^j (p+n)[p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}|}{(p+j+1)(p+j+1-\alpha|u_{p+n}|)} \quad (4.6)$$

and

$$C_j = \frac{p^2-\alpha p|u_p| - \sum_{n=1}^j (p+n)[p+n-u_{p+n}|+(1-\alpha)|u_{p+n}|]|a_{p+n}|}{(p+j+1)(p+j+1-\alpha|u_{p+n}|)} \quad (4.7)$$

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