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COINCIDENCE THEOREM OF SYSTEM OF MAPS ON PRODUCT SPACES

Ritu Arora*

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ABSTRACT

The purpose of this paper is to obtain coincidences theorem for four systems of Matkowski type maps on a finite product of metric spaces.

Key words and phrases: *Coincidence point, fixed point*

INTRODUCTION

In 1976, Jungck [9] obtained a common fixed-point theorem for commuting maps generalizing the Banach's fixed point theorem. This result was first generalized by Singh in 1997. Jungck's result [op.cit.] and Singh's generalizations [op.cit.] gave a new direction of work in contractive fixed point theory. Consequently, numerous extensions, generalizations and applications were proposed (See, for instance, [5], [7], [8], [15], [16], [18], [19], [20], and several references thereof).

In 1973, Matkowski [12] (see also [13]) obtained a new contraction principle, in fact an extension of well known Banach Contraction Principle, for a system of maps on a finite product of metric spaces (see theorem M below). Theorem M (generally called the Matkowski contraction principle or briefly Mcp) has been studied on various setting, and its extensions and generalization encompass several general result from contractive fixed point theory (see for instance [1], [2], [3], [4], [10], [14], [15], [19], [21]). Matkowaski type fixed point theorem are applicable in solving abstract equations on product spaces to convex solutions of a system of functional equations and other abstract equations (cf. [1], [2], [12], [15], [16]).

Kominek [11]and Singh et al. [19] independently unified apparently diverse development of Jungck type contractive maps and Matkowaski type maps in a natural way. Singh et al. [18] introduced weakly commuting and asymptotically commuting (also called compatible maps) for Jungck Matkowaski type maps on product spaces. This paved the way to study hybrid type maps on these setting as well (see [1], [2] and [7]).

In this paper, we obtain coincidence theorems for four systems Jungck-Matkowaski type maps on a product of n metric spaces on coincidence for such maps.

PRELIMINARIES

We shall follow the following notations and definitions from Matkowski [12]-[13] (see also Czerwik [3], and Singh et al. [18] [19] [20] & [21]).

Let a_{ik} be non negative numbers, $i, k = 1,2, \dots, n,$ *and* C_{ik} square matrix defined in the following recursive manner.

$$C_{ik}^{0} = \begin{cases} a_{ik}, & for\ i \neq k \\ 1 - a_{ik}, & for\ i = k \end{cases} \qquad (2.1)$$

$i, k = 1,2, \dots, n,$ *and* C_{ik}^{t} *are defined recursively by*

$$C_{ik}^{(t+1)} = \begin{cases} C_{11}^{(t)}\, C_{i+1,k+1}^{(t)} + C_{i+1,1}^{(t)}\, C_{1,\ k+1}^{(t)} & for\ i \neq k \\ C_{11}^{(t)}\, C_{i+1,k+1}^{(t)} + C_{i+1,1}^{(t)}\, C_{1,\ k+1}^{(t)} & for\ i = k, \end{cases} \qquad (2.2)$$

$$i\,, k = 1\,, \dots, n - t - 1, t = 0,1,2 \dots, n - 2. if\ n = 1, we\ define\ C_{11}^{(0)} = a_{11}$$

The following Lemma is essentially due to Matkowaski [15, p9] (see also [12], [15], [16], [19]).

Corresponding author: Ritu Arora*

Lemma: Let $c_{ik}^{(0)} > 0$, i, k = 1, 2….., n, n ≥ 2 then the system of inequalities

$$\sum_{k=1}^{n} a_{ik} r_k < r_i, \quad i, k = 1, 2 \ldots\ldots, n. \tag{2.3}$$

has a solution $r_i > 0, \quad i = 1, 2 \ldots\ldots, n$, if and only if

$$c_{ii}^{(t)} > 0, \quad i = 1, 2 \ldots\ldots, n - t, \quad t = 0, 1, \ldots\ldots, n-1 \tag{2.4}$$

hold. Indeed, there exists a number h ∈ (0, 1) such that

$$\sum_{k=1}^{n} a_{ik} r_k \leq h r_i, \quad i, k = 1, 2 \ldots\ldots, n, \tag{2.5}$$

for some positive numbers $r_1, r_2, \ldots\ldots\ldots, r_n$, (see Matkowski [23], Kominek[19]). Indeed such an h may be obtained (cf. [6]) by

$$h = \max_i \left\{ r_i^{-1} \sum_{k=1}^{n} a_{ik} \, r_k \right\} \tag{2.6}$$

Let $(X_i, d_i), \; i = 1, 2, \ldots\ldots, n$, be metric spaces,

$$X := X_1 \times X_2 \times \ldots\ldots \times X_n,$$

and

$$x^m := (x_1^m, \ldots.., x_n^m), \; x_i^m \in X_i, \; i = 1, 2, \ldots\ldots, n \; ; \; m = 0, 1, 2 \ldots\ldots$$

Also,

$$x := (x_1, \ldots.., x_n), \; x_i \in X_i, \; i = 1, 2, \ldots\ldots, n \; .$$

Thus $y \in X \;\; means \;\; y := (y_1, \ldots.., y_n)$.

Theorem M ([12]-[13]): Let $(X_i, d_i), \; i = 1, 2, \ldots\ldots, n$ be complete metric spaces and $T_i : X \rightarrow X_i, \; i = 1, 2, \ldots\ldots, n$, such that

$$d_i(T_i x, T_i y) \leq \sum_{k=1}^{n} a_{ik} d_k(x_k, y_k), \; i = 1, 2, \ldots\ldots, n, \tag{2.7}$$

for every $x_k, y_k \in X_k, \; i, k = 1, 2, \ldots\ldots, n$ where a_{ik} are nonnegative numbers such that the matrices defined in (2.1) and (2.2) satisfy the condition (2.3). Then the system of equations

$$T_i x = x_i, \; i = 1, 2, \ldots\ldots, n, \tag{2.8}$$

has exactly one solution $p := (p_1, \ldots.., p_n)$ such that $p_i = X_i, \; i = 1, 2, \ldots\ldots, n.$ .Further, for any $x^0 \in X$, the sequence of successive approximations

$$x_i^{(m+1)} = T_i \, x^m, \; i = 1, 2, \ldots\ldots, n, \; m = 0, 1, 2, \ldots\ldots \tag{2.9}$$

converges and

$$p_i = \lim_{m \to \infty} x_i^m, \; i = 1, 2, \ldots\ldots, n \; . \tag{2.10}$$

The system of maps $(T_1, \ldots\ldots\ldots, T_n)$ (which henceforth, will be denoted by (T_i) on X with values in metric spaces X_i , $i = 1, 2, \ldots\ldots\ldots, n$, satisfying (2.7) may be called the system of contractions on a product of metric spaces (or simply Matkowski Contraction on product of spaces), wherein (2.1) - (2.3) hold.

Notice that Theorem M with n = 1 is the well known Banach Contraction Principle (BCP). We remark that Theorem M becomes equivalent to BCP if the Matrix (a_{ik}) is symmetric (see Matkowski and Singh [24, cor 2]).

COINCIDENCE THEOREM

Theorem: Let $A_1, \ldots, A_n$ be an arbitrary nonempty sets and A: = A_1 x A_2 x … x A_n. Let (X_1, d_1), … , (X_n, d_n) be the metric spaces and P_i, Q_i, S_i, T_i maps defined on A with values in X_i, i = 1, …, n, such that

$$P_i(A) \cup Q_i(A) \subset S_i(A) \cap T_i(A), \ \ i = 1, 2, \ldots\ldots, n; \tag{3.1}$$

One of P_i(A) or Q_i(A) or $S_i(A) \cap T_i(A)$ is a complete subspace of X_i, i = 1, 2, … , n (3.2)

$$d_i(P_i\, x,\, Q_i\, y) \leq \max\left\{\sum_{k=1}^{n} a_{ik}\, d_k(S_k\, x, T_k\, y), b\, \max\{d_i(S_i\, x, P_i\, x),\, d_i(T_i\, y, Q_i\, y), \frac{1}{2}\{d_i(S_i\, x, Q_i\, y) + d_i(T_i\, y, P_i\, x)\}\,]\right\} \tag{3.3}$$

$forall\, x,\ y\ in\, A,\ where\ a_{ik}\ \ and\ \ b$ are non negative numbers, b < 1 and (2.1), (2.2) and (2.3) hold. Then the system of equations

$$P_i\, x = S_i\, x = x_i = Q_i\, x = T_i\, x,\ \ i = 1, 2, \ldots\ldots\ldots, n, \tag{3.4}$$

has a solution in A, that is

i) The systems (P_i) and (S_i) have a coincidence,
ii) (Q_i) and (T_i) have (possibly a different) coincidence, and
iii) Their coincidence values are the same.

Proof: First we note that system (2.3) and (2.4) are equivalent. From the above Lemma we may choose a system of positive numbers $r_{1,\ldots,} r_n$ such that

$$\sum_{k=1}^{n} a_{ik}\, r_k \leq h r_i \quad ,i = 1, 2, \ldots, n \tag{3.5}$$

Pick $x_i^{(0)} \in X_i, i = 1, \ldots, n.$ We, in view of(), construct a sequences $\{x_i^m\} and \{z_i^m\} in S_i(A) \cap T_i(A)$

$$P_i x^{2m} = T_i x^{2m+1} := Z_i^{2m+1}$$
$$and$$
$$Q_i x^{2m+1} = S_i x^{2m+2} := Z_i^{2m+2} m = 0, 1, 2, \ldots.$$

From the homogeneity of the system (3.4), we may assume (indeed, if necessary, increasing the value of r_i, i = 1,…,n.)

such that

$d_i(z_i^1, z_i^2) \leq r_i$, i = 1,…, n

$Z_i^m \neq Z_i^{m+1}$, i = 1,…, n

$d_i(z_i^2, z_i^3) = d_i(P_i x^2, Q_i x^1)$

$$\leq \max \sum_{k=1}^{n} a_{ik}\, d_k(S_k\, x^2, T_k\, y^1), b \max\{d_i(S_i\, x^2, P_i\, x^2), d_i(T_i\, x^1, Q_i\, x^1), \frac{1}{2}\{d_i(S_i\, x^2, Q_i\, x^1) + d_i(T_i\, x^1, P_i\, x^2)\}]\}$$

$$= \max\{\sum_{k=1}^{n} a_{ik}\, d_k(z_k^2, z_k^1), b \max[d_i(z_i^2, z_i^3), d_i(z_i^1, z_i^2), \frac{1}{2}\{d_i(z_i^1, z_i^3)$$

$$\leq \max\{h\, r_i, b\, d_i(z_i^2, z_i^3), b\, d_i(z_i^2, z_i^1), \frac{b}{2}[d_i(z_i^1, z_i^3) + d_i(z_i^2, z_i^3)]\}$$

That is

$$d_i(z_i^2, z_i^3) \leq \max\{hr_i, bd_i(z_i^1, z_i^2)\}$$

Therefore

$$d_i(z_i^2, z_i^3) \leq \max\{h, b\} r_i$$
$$\leq cr_i, where c = \max\{h, b\}.$$

Now considering

$d_i(z_i^3, z_i^4) = d_i\{P_i x^2, Q_i x^3\}$, Proceeding as above,

We see that

$$d_i(z_i^3, z_i^4) \leq c^2 r_i$$

Inductively

$$d_i(z_i^{(m+1)}, z_i^{(m+2)}) \leq c^m r_i, m = 1, 2, ...$$

Therefore $\{z_i^m\}$ is a Cauchy sequence. if $S_i(A) \cap T_i(A)$ is a complete subspace of X_i, then $\{z_i^m\}$ has a limit u_i (say) in $S_i(A) \cap T_i(A)$, i = 1, …, n.

Now let V_i be a point in $S_i^{-1} u_i$ that is

$$S_i V_i = u_i, \; i = 1, 2, \ldots, n \qquad (3.6)$$

Note that if one of $P_i(A)$ or $Q_i(A)$ is a complete subspace of X_1 then, the in view of (3.1) the relation(3.5) hold obviously.

Then from (3.3)

$$d_i(S_i\, V_i, P_i V_i) \leq d_i(S_i\, V_i, Q_i\, x^{(2m+2)}), d_i(P_i\, V_i, Q_i\, x^{2m+1}),$$

$$\leq d_i(S_i\, V_i, Q_i\, x^{(2m+2)}), \max\{\sum_{k=1}^{n} a_{ik} d_k(S_k\, V_k, Z_k^{(2m+1)}), b \max\{d_i(S_i\, V_i, P_i V_i), d_i(Z_i^{(2m+1)}, Z_i^{(2m+2)}),$$
$$\frac{1}{2}\{d_i(S_i\, V_i, Z_i^{(2m+2)}) + d_i(Z_i^{(2m+1)}, P_i\, V_i)]\}$$

Making $m \to \infty$, we get,

$$d_i(S_i V_i, P_i V_i) \leq bd_i(S_i V_i, P_i V_i),$$

This yields

$$S_iV_i = P_iV_i = u_i, \; i = 1, \ldots, n.$$

Similarly there exists a point W_i in $T_i^{-1} u_i$ such that

$$T_iW_i = Q_iW_i = u_i, \; i = 1, \ldots, n.$$

This completes the proof.

REMARKS

1. The coincidence part of the Jungck contraction principle (see [16]) is obtained from Theorem 1 when $b = 0$, $A_i = X_i$, $P_i = Q_i$ and $S_i = T_i$, $n = 1$.

2. Mcp (Theorem M) is obtained as a corollary from Theorem 1 when $b = 0$, $A_i = X_i$, $P_i = Q_i$ and $P_ix = S_ix = x_i$ for each x_i in X_i, $i = 1\ldots n$.

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