International Journal of Mathematical Archive-3(6), 2012, 2457-2465
IJMA Available online through www.ijma.info ISSN 2229 – 5046

STABILITY OF FUNCTIONAL EQUATIONS IN 2-NORMED LINEAR SPACES

Sushma Lather* & Anil Kumar**
***Kanya Mahavidyalaya Kharkhoda, Sonepat, Haryana, India**
****Unique Institute of Education, Rohtak, Haryana, India**

(Received on: 16-05-12; Accepted on: 31-05-12)

ABSTRACT

In this paper I prove the stability of Cauchy functional equations in 2-normed linear spaces and also deal with Jensen type function equations and their stability in 2-normed linear spaces.

Keywords: *Cauchy functional equation, Jensen functional equation, stability, 2-norm linear space.*

2000 Mathematics Subject Classification: *Primary 54E40; Secondary 39B82, 46S50, 46S40.*

INTRODUCTION

The concept of linear 2-normed spaces has been investigated by S.Gahler [6] in 1964 and has been developed extensively in different subjects by many authors.

Definition1.1: Let X be a linear space of dimension greater than 1. Suppose $\|\bullet,\bullet\|$ is a real-valued function on $X \times X$ satisfying the following conditions:

1. $\|x, y\| = 0$ if any only if x, y are linearly dependent vectors,
2. $\|x, y\| = \|y, x\|$ for all $x, y \in X$,
3. $\|\lambda x, y\| = |\lambda|\, \|x,y\|$ for all $\lambda \in R$ and for all $x, y \in X$,
4. $\|x+y, z\| \le \|x, z\| + \|z, y\|$ for all $x, y, z \in X$.

Then $\|\bullet,\bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet,\bullet\|)$ is called 2-normed linear space. Some of the basic properties of 2-norm are that they are non-negative and $\|x, y+\lambda x\| = \|x, y\|$ for all $\lambda \in R$ and for all $x, y \in X$.

1.2 Examples of 2-normed linear spaces

- Vector space R^2 is a 2-normed space with respect to the following 2-norm

$\|x, z\| = \{((x_1)^2+(x_2)^2)((z_1)^2+(z_2)^2) - (x_1z_1+x_2z_2)^2\}$.

- Vector space R^3 is a 2-normed space with respect to the following 2-norms

(1) $\|x, y\|_1 = \max\{|x_1y_2 - x_2y_1| + |x_1y_3 - x_3y_1|, |x_1y_2 - x_2y_1| + |x_2y_3 - x_3y_2|\}$, where $x, y \in R^3$
(2) $\|x, y\|_2 = \max\{|x_1y_2 - x_2y_1|, |x_1y_3 - x_3y_1|, |x_2y_3 - x_3y_2|\}$, where $x, y \in R^3$
(3) $\|x, y\|_3 = \frac{1}{2}\{\max\{|x_1y_2 - x_2y_1|, |x_1y_3 - x_3y_1|, |x_2y_3 - x_3y_2|\}$, where $x, y \in R^3$

2. FUNCTIONAL EQUATIONS

A functional equation meaning an equation between functionals: an equation $F = G$ between functional can be read as an 'equation to solve', with solutions being themselves functions. In such equations there may be several sets of variable unknowns, like when it is said that an additive function *f* is one satisfying the functional equation

$$f(x+y) = f(x)+f(y)$$

In other words, a functional equation is an equation whose variables are ranging over functions.

***Corresponding author:** *Sushma*, **Kanya Mahavidyalaya Kharkhoda, Sonepat, Haryana, India*

2.1 Stability problem and stability of Cauchy functional equation

A question in the theory of functional equations is the following "When is it true that a function which approximately satisfies a functional equation $\in$ must be close to an exact solution $\in$?" If the problem accepts a solution, we say that the equation $\in$ is stable.

In 1940, S.M. Ulam [18] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphism.

Theorem 2.1 [18]: Let (G_1, *) be a group and (G_2, $\circ$, d) be a metric group with the metric d. Given $\in >0$, does there exists a $\delta_\in >0$ such that if a mapping h: $G_1 \rightarrow G_2$ satisfies the inequality d (h(x*y),h(x)$\circ$ h(y))< $\delta_\in$ $\forall$ x, y$\in G_1$, then there is a mapping H: $G_1 \rightarrow G_2$ such that for each x, y $\in G_1$ H(x*y)=H(x) $\circ$H(y) and d(h(x),H(x))< $\in$?

In the next year, D.H. Hyers [9], gave answer to the above question for additive groups under the assumption that groups are Banach spaces. In 1978, T.M. Rassias [16] proved a generalization of Hyers' theorem for additive mapping as a special case in the form of following result.

Theorem 2.2[16]: Suppose that E and F are real normed spaces with F a complete normed space, f: E$\rightarrow$F is a mapping such that for each fixed x$\in$E the mapping t$\rightarrow$f (tx) is continuous on R, and let there exist $\in \geq 0$ and

p$\in$[0,1) s.t $\|f(x+y)-f(x)-f(y)\| \leq \varepsilon(\|x\|^p+\|y\|^p)$ x, y$\in$E.

Then there exists a unique linear mapping T: E$\rightarrow$F s.t $\|f(x)-T(x)\| \leq \varepsilon \dfrac{\|x\|^p}{(1-2^{p-1})}$, x$\in$E

3. MAIN RESULTS

I generalize the result of Rassias in 2-normed linear spaces as follows:

Theorem 3.1: Suppose f: X$\rightarrow$Y is a mapping where X is 2-normed spaces and Y a complete normed space and for some $\in \geq 0$ and p$\in$[0,1) such that

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon(\|x,z\|^p+\|y,z\|^p) \quad \forall \text{ x, y, z} \in \text{X}. \tag{1}$$

Then there exists a unique linear mapping A: X$\rightarrow$Y such that

$$\|f(x)-T(x)\| \leq \varepsilon \frac{\|x,z\|^p}{(1-2^{p-1})} \tag{2}$$

for all x, z$\in$X.

Proof: If we put y=x in (1), we get the following inequality

$$\|f(2x)-2f(x)\| \leq 2\varepsilon\|x,z\|^p$$

For all x, z$\in$X. we replace x by 2^{k-1}x (for k$\in$N), we obtain

$$\|f(2^k x)-2f(2^{k-1}x)\| \leq 2^{kp-p+1}\varepsilon\|x,z\|^p$$

Multiplying both sides of the above inequality by $\dfrac{1}{2^k}$ and then adding the resulting n inequalities, we get

$$\sum_{k=1}^{n}\frac{1}{2^k}\|f(2^k x)-2f(2^{k-1}x)\| \leq \varepsilon\|x,z\|^p \sum_{k=1}^{n}\frac{2^{2k-p+1}}{2^k}.$$

Using the triangle inequality

$$|\|a+b\| \le \|a\| + \|b\|$$

And simplifying the left side of the inequality, we get

$$\left\|\frac{1}{2^n} f(2^n x) - f(x)\right\| \le \varepsilon \|x, z\|^p \sum_{k=1}^{n} 2^{k(p-1)}.2^{1-p}. \tag{3}$$

Since

$$\sum_{k=1}^{n} 2^{k(p-1)} \le \sum_{k=1}^{\infty} 2^{k(p-1)}.$$

The inequality (2) yields

$$\left\|\frac{1}{2^n} f(2^n x) - f(x)\right\| \le \varepsilon \|x, z\|^p \, 2^{1-p} \sum_{k=1}^{\infty} 2^{k(p-1)}.$$

which is

$$\left\|\frac{1}{2^n} f(2^n x) - f(x)\right\| \le \frac{2\varepsilon}{2-2^p} \|x, z\|^p. \tag{4}$$

For all x, z $\in$ X. By induction it can be shown that (4) is valid for all natural numbers. If m > n > 0, then m-n is a natural number and replacing n by m-n in (4), we get

$$\left\|\frac{1}{2^{m-n}} f(2^{m-n} x) - f(x)\right\| \le \frac{2\varepsilon}{2-2^p} \|x, z\|^p$$

which is

$$\left\|\frac{1}{2^m} f(2^{m-n} x) - \frac{1}{2^n} f(x)\right\| \le \frac{1}{2^n} \cdot \frac{2\varepsilon}{2-2^p} \|x, z\|^p \tag{5}$$

For all x, z $\in$ X. Replacing x by $2^n x$ in (5), we obtain

$$\left\|\frac{1}{2^m} f(2^m x) - \frac{1}{2^n} f(2^n x)\right\| \le \frac{2^{np}}{2^n} \cdot \frac{2\varepsilon}{2-2^p} \|x, z\|^p \tag{6}$$

Since $0 \le p < 1$,

$$\lim_{n\to\infty} 2^{n(p-1)} = 0$$

And hence from (6), we obtain

$$\lim_{n\to\infty} \left\|\frac{1}{2^m} f(2^m x) - \frac{1}{2^n} f(2^n x)\right\| = 0.$$

Therefore $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence. This Cauchy sequence has a limit in Y. we define

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \quad \text{for all } x \in X \tag{7}$$

First we claim that A: $X \to Y$ is additive mapping. Now, we consider

$$\begin{aligned}\|A(x+y) - A(x) - A(y)\| &= \lim_{n\to\infty}\frac{1}{2^n}\| f(2^n x+2^n y) - f(2^n x) - f(2^n y)\| \\ &\le \lim_{n\to\infty}\frac{\varepsilon\left(\|x,z\|^p + \|y,z\|^p\right)2^{np}}{2^n} \qquad \text{(by (6))} \\ &= 0,\end{aligned}$$

Since $p\in[0, 1)$

Hence $A(x + y) = A(x) + A(y)$, for all $x, y\in X$. Now again consider

$$\begin{aligned}\|A(x) - f(x)\| &= \left\|\lim_{n\to\infty}\frac{f(2^n x)}{2^n} - f(x)\right\| \\ &= \lim_{n\to\infty}\left\|\frac{f(2^n x)}{2^n} - f(x)\right\| \\ &\le \lim_{n\to\infty}\frac{2\varepsilon}{2-2^p}\|x,z\|^p.\end{aligned}$$

Hence we get

$$\|A(x) - f(x)\| \le \frac{2\varepsilon}{2-2^p}\|x,z\|^p \quad \text{for all } x, z \in X.$$

Now, we show that A is unique.

Suppose A is not unique. Then there exists another additive mapping B: $X \to Y$ such that

$$|B(x) - f(x)| \le \frac{2\varepsilon}{2-2^p}\|x,z\|^p$$

Hence

$$\begin{aligned}\|B(x) - A(x)\| &\le \|B(x) - f(x)\| + \|A(x) - f(x)\| \\ &\le \frac{2\varepsilon}{2-2^p}\|x,z\|^p + \frac{2\varepsilon}{2-2^p}\|x,z\|^p \\ &= \frac{4\varepsilon}{2-2^p}\|x,z\|^p.\end{aligned}$$

Further, since A and B are additive, we have

$$\begin{aligned}\|A(x) - B(x)\| &= \frac{1}{n}|A(nx) - B(nx)| \\ &\le \frac{1}{n^{1-p}}\frac{4\varepsilon}{2-2^p}\|x,z\|^p \qquad (8)\end{aligned}$$

Hence taking limit as $n\to\infty$ on both sides, we get from (8)

$$\lim_{n\to\infty}\|A(x) - B(x)\| \le \lim_{n\to\infty}\frac{1}{n^{1-p}}\frac{4\varepsilon}{2-2^p}\|x,z\|^p$$

Hence

$$\|A(x) - B(x)\| \le 0$$

Therefore $A(x) = B(x)$, for all $x\in R$. Hence A is unique.

3.2 Stability of Jensen functional equations.

The functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$$

is known as Jensen's equation.. By replacing x and y by x+y and x-y respectively, Jensen equation can be written as

$$f(x+y) + f(x-y) = 2 f(x).$$

The first result on the stability of Jensen's equation.was obtained by Z. Kominek [12] in 1989. He proved the following result:

Theorem 3.3 [12]: Let D be a subset of R^n with non-empty interior. Assume that there exists an x_0 in the interior of D such that $D_0 = D - x_0$ satisfies the condition $(1/2)D_0 \subset D_0$. Let a mapping f: D $\rightarrow$ Y satisfy the inequality

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\le\delta,$$

for some $\delta \geq 0$ and for all x,y $\in$ D. then there exist a mapping F: $R^n \rightarrow$ Y and a constant K>0 such that

$$2F\left(\frac{x+y}{2}\right)=F(x)+F(y)$$

for all x, y $\in R^n$, and $\|f(x)-F(x)\| \leq K$ for all $x \in D$.

In 1998, S.M. Jung [10] generalized the Hyers-Ulam- Rassias stability of Jensen's equation and its applications. He proved the stability of Jensen's functional equation by using the concepts of Th. M. Rassias [16] and D. H. Hyers [9], i.e. the stability of the functional inequality

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\le\delta+\theta\,(\|x\|^p+\|y\|^p)$$

for the case $p \geq 0$ $(p \neq 1)$.

Now, we generalized the above result in 2-normed linear spaces as follows.

The Jensen's functional inequality

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\le\delta+\theta\,(\|x,z\|^p+\|y,z\|^p) \qquad (9)$$

For the case $p \geq 0$ $(p \neq 1)$.

Theorem 3.5: Let p > 0 be given with $p \neq 1$. Suppose a mapping f : X $\rightarrow$ Y, where X is a 2-normed linear space and Y is a Banach space, satisfies the inequality (9) for all x, y, $z \in X$. Further assume $f(0) = 0$ and $\delta = 0$ in (9) for the case of p > 1. Further suppose that z is not in linear span of x.

Then there exists a unique additive mapping F: X $\rightarrow$ Y such that

$$\|f(x)-F(x)\|\le\delta+\|f(0)\|+\frac{\theta}{2^{1-p}-1}\|x,z\|^p, \quad \text{(for } p<1) \qquad (10)$$

or

$$\|f(x)-F(x)\|\le\frac{2^{p-1}\theta}{2^{p-1}-1}\|x,z\|^p, \qquad \text{(for } p>1)$$

for all x, $z \in X$.

Proof: If we put y = 0 in (9), then we get the following inequality,

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \leq \delta + \| f(0) \| + \theta \|x, z\|^p, \tag{11}$$

for all x,z∈X . By taking induction on n, we show that

$$\| 2^{-n} f(2^n x) - f(x) \| \leq (\delta + \| f(0) \|) \sum_{k=1}^{n} 2^{-k} + \theta \|x, z\|^p \sum_{k=1}^{n} 2^{-(1-p)k} \tag{12}$$

for the case when 0 < p < 1. By substituting 2x for x in (11) and dividing by 2 both sides of (11) we see the validity of (12) for n = 1. Now, we consider that the inequality (6.4.4) holds for n∈N. Now replace x in (11) by 2^{n+1} x and dividing both sides of (11) by 2, then it follows from (12) that

$$\begin{aligned} \| 2^{-(n+1)} f(2^{n+1} x) - f(x) \| &\leq 2^{-n} \| 2^{-1} f(2^{n+1} x) - f(2^n x) \| + \| 2^{-n} f(2^n x) - f(x) \| \\ &\leq (\delta + \| f(0) \|) \sum_{k=1}^{n+1} 2^{-k} + \theta \|x, z\|^p \sum_{k=1}^{n+1} 2^{-(1-p)k} \end{aligned}$$

This completes the proof of inequality (12).

Now we define

$$F(x) = \lim_{n \to \infty} 2^{-n} f(2^n x). \tag{13}$$

for all x∈X. we now prove that the sequence { $2^{-n} f(2^n x)$ } is a Cauchy sequence for all x∈X. For n > m we use (12) to get

$$\begin{aligned} \| 2^{-n} f(2^n x) - 2^{-m} f(2^m x) \| &= 2^{-m} \| 2^{-(n-m)} f(2^{n-m} 2^m x) - f(2^m x) \| \\ &\leq 2^{-m} \left(\delta + \| f(0) \| + \frac{2^{mp}\theta}{2^{1-p} - 1} \|x, z\|^p\right) \\ &\to 0 \text{ as } m \to \infty \end{aligned}$$

Let x, z∈X be arbitrary. It follows from (13) and (9) that

$$\begin{aligned} \| F(x+y) - F(x) - F(y) \| &= \lim_{n \to \infty} 2^{-(n+1)} \left\| 2f\left(\frac{2^{n+1}(x+y)}{2}\right) - f(2^{n+1} x) - f(2^{n+1} y) \right\| \\ &\leq \lim_{n \to \infty} 2^{-(n+1)} [\delta + 2^{(n+1)p} \theta (\|x, z\|^p + \|y, z\|^p)] \\ &= 0 \end{aligned}$$

Hence, F is an additive mapping and the inequality (12) and the definition (13) imply the validity of (10).

Now, let G: X → Y be another additive mapping which satisfies the inequality (10). Then, it follows from (10) that

$$\begin{aligned} \| F(x) - G(x) \| &= 2^{-n} \| F(2^n x) - G(2^n x) \| \\ &\leq 2^{-n} (\| F(2^n x) - f(2^n x) \| + \| f(2^n x) - G(2^n x) \|) \\ &\leq 2^{-n} \left(2\delta + 2\| f(0) \| + \frac{2^{np+1}\theta}{2^{1-p} - 1} \|x, z\|^p\right), \end{aligned} \tag{14}$$

for all x∈X and for any n∈N. Since the right hand side of (14) tends to 0 as n → ∞, we conclude that F(x) = G(x), for all x∈X, which proves the uniqueness of F. For the case p>1 and δ = 0 in the functional inequality (9) we can analogously prove the inequality

$$\| 2^n f(2^{-n} x) - f(x) \| \leq \theta \|x, z\|^p \sum_{k=0}^{n-1} 2^{-(p-1)k}$$

instead of (12). The rest of the proof for this case goes through in the similar way.

Remark 6.4.3: Let $p\in[0, 1)$ be given. By substituting x + y for x and putting y = 0 in (9), we get

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x+y)\right\| \le \delta + \|f(0)\| + \theta\ (\|x,z\|^p+\|y,z\|^p)$$

This inequality, together with (9), gives

$$\|f(x+y)-f(x)-f(y)\| \le 2\delta + \|f(0)\| + 2\theta\ (\|x,z\|^p+\|y,z\|^p),$$

for all x, y, z∈X. then there exists a unique additive mapping F: X → Y such that

$$\|f(x)-F(x)\| \le 2\delta + \|f(0)\| + \frac{2\theta}{1-2^{p-1}}\|x,z\|^p,\ x, z\in X.$$

Theorem 3.6: Let d > 0 and δ ≥ 0 be given. Assume that a mapping f : X → Y satisfies the functional inequality

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \le \delta, \tag{15}$$

for all x, y∈X with $\|x,z\|+\|y,z\| \ge$ d. Then there exists a unique additive mapping F: X→Y such that

$$\|f(x)-F(x)\| \le 5\delta + \|f(0)\|,\quad \text{for all } x\in X. \tag{16}$$

Proof: Suppose $\|x,z\|+\|y,z\|$ < d. If x = y = 0, we can choose a w∈X such that $\|w\|$=d. Otherwise, let w = (1 + d/ $\|x,z\|$) x for $\|x,z\| \ge \|y,z\|$ or w = (1 + d/$\|y,z\|$) y for $\|x,z\| < \|y,z\|$. It is then obvious that

$$\begin{aligned}
&\|x-w,z\|+\|y+w,z\| \ge d;\\
&\|2w,z\|+\|x-w,z\| \ge d;\\
&\|y,z\|+\|2w,z\| \ge d;\\
&\|y+z\|+\|z\| \ge d;\\
&\|x\|+\|z\| \ge d.
\end{aligned} \tag{17}$$

Form (15) and (17) and the relation

$$\begin{aligned}
2f\left(\frac{x+y}{2}\right)-f(x)-f(y) &= 2f\left(\frac{x+y}{2}\right)-f(x-z)-f(y+z)\\
&\quad -\left[2f\left(\frac{x+z}{2}\right)-f(2z)-f(x-z)\right]\\
&\quad +\left[2f\left(\frac{y+2z}{2}\right)-f(y)-f(2z)\right]\\
&\quad -\left[2f\left(\frac{y+2z}{2}\right)-f(y+z)-f(z)\right]\\
&\quad +\left[2f\left(\frac{x+z}{2}\right)-f(x)-f(z)\right]
\end{aligned}$$

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \le 5\delta \tag{18}$$

Thus mapping f satisfies the inequality (15) for all x, y,z ∈X. Therefore, there exists a unique additive mapping F: X → Y which satisfies the inequality (16) for all x∈ X.

Corollary 3.7: Suppose a mapping f : X $\to$ Y satisfies the condition f (0) = 0 (X having 2-norm structure). Also f satisfies the following asymptotic condition

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \to 0$$

as $\|x,z\| + \|y,z\| \to \infty,$ (19)

for a fixed z in X, with z not being in the linear span of x and y, then f is an additive mappings and converse of this proposition holds.

Proof: On account of (18), there exists a sequence (δ_n), monotonically decreasing to 0, such that

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \delta_n, \tag{20}$$

for all x, y$\in$X with $\|x,z\| + \|y,z\| \geq$ n. It then follows from (20) and by above Theorem 3.7 there exists a unique additive mapping F_n: X $\to$ Y such that

$$\| f(x) - F_n(x) \| \leq 5\delta_n, \tag{21}$$

for all x$\in$ X. Let l, m$\in$N satisfying m $\geq$ l. obviously, it follows from (21) that

$$\| f(x) - F_m(x) \| \leq 5\delta_m \leq 5\delta_l,$$

for all x$\in$X, since (δ_n) is a monotonically decreasing sequence. The uniqueness of F_n implies $F_m = F_l$. Hence, by letting n $\to \infty$ in (21). We conclude that f is additive. The reverse assertion is trivial. Hence proved.

REFERENCES

[1] Belaid, B., On the Hyers – Ulam – Rassias Stability of the Cauchy Linear Functional Equation, Tamsui Oxford J. of Math. Sci. 23(4) (2007) 449-459.

[2] Bouikhalene, B. and Elquorachi, E., Ulam-Gavruta-Rassias stability of a Pexider functional equation, Euler's FIDA in IJAMAS, vol.1 (2007).

[3] Brzdek, J., On a method of proving the Hyers- Ulam stability of functional equations on restricted domains, The Australian Journal of Mathematical Analysis and Applications, 6(2009) 1-10.

[4] Cerone, P., Cho, Y.J., Dragomir, S.S. and Kim, S.S., Refinements of some reverses of Schwarz's inequality in 2-inner product spaces and applications for integrals, arXiv: math/0309117v1 [math.FA] 6 Sep 2003.

[5] Chang, I.S., Jun, K.M. and Jung, Y.S., The modified Hyers-Ulam-Rassias stability of cubic type functional equation, Math. Inequa. Appl. 8(2005)pp. 675-683.

[6] G¨*a*hler, S., Lineare 2-Normierte R¨aume, Math. Nachr., **28**(1964) 1-43.

[7] Gahler, S., 2 Metrische Raume und ihre Topologische Struktur, Math. Nachr., 26(1963), 115-148.

[8] Hyers, D.H. and Rassias, Th. M., Approximate Homomorphisms, Aequationes Math., 44 (1992), 125–153.

[9] Hyers, D.H., On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A., 27(1941) 222-224.

[10] Jung, S.M., Hyers-Ulam-Rassias Stability of Jensen's Equation and its Application, Proc. Amer. Math. Soc., 126 (1998), 3137–3143.

[11] Jung, S.M., On the Hyers-Ulam stability of functional equation that has the quadratic property, J. Math. Anal. Appl., 222(1998)126-137.

[12] Kominek, Z. On a Local Stability of the Jensen Functional Equation, Demonstration Math. 22(1989), 499-507.

[13] Rassias, T.M. and Semrl, P., On the behavior mapping which do not satisfy Hyer's Ulam stability, Proc. Amer. Math. Soc. 114(1992) 989-993.

[14] Rassias, T.M. On the Stability of Functional Equations and a Problem of Ulam, Acta Appl. Math., 62(2000), 123-130.

[15] Rassias, T.M., On the Stability of Functional Equations in Banach spaces, J. Math. Anal. Appl., 251 (2000), 264–284.

[16] Rassias, T.M., Problem 16; 2, Report of the 27th International Symp. on Functional Equations, Aequationes Math., 39 (1990), 292–293; 309.

[17] Ulam, S.M., A collection of Mathematical Problems, Interscience tracts in Pure and Applied Mathematics, U.S.A 16(1960).

[18] Ulam, S.M., A Collection of the Mathematical Problems, Interscience Publ. New York, 1960.

[19] Ulam, S.M., Problems in Modern Mathematics, John Wiley & Sons, New York, USA, 1964.

Source of support: Nil, Conflict of interest: None Declared