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# A GENERALIZED COMMON FIXED POINT THEOREM FOR SET VALUED MAPPINGS

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***ABSTRACT***

*In this paper we generalize a common fixed point theorem which properly generalized the theorem of Ahmed [1], and theorem of Itoh and Takahashi [3] , and extend the theorems of Kasahara and Rhoades [7], Tas , Telei and Fisher [9] and Telei , Tas and Fisher [10] of the set valued mapping.*

***KEY WORDS***: *Common Fixed Point, Compact Metric Space, Weakly Compatible Mappings, Set Valued Mapping.*

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## 1. INTRODUCTION

Let (X, d) be a metric space and B(X) the set of all non empty bounded subsets of X. Let for all A, B $\in$ B(X), $\delta$(A, B) and D (A, B) be the functions defined by

$$\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}$$
$$D(A, B) = \inf \{d(a, b) : a \in A, b \in B\}$$

If $\quad A = \{a\}, \quad \delta(A, B) = \delta(a, B).$

If $\quad B = \{b\}$ also, $\quad \delta(A, B) = d(A, b).$

For all A, B, C $\in$ B(X), it follows immediately from the definition that

$$\delta(A, B) = \delta(B, A) \geq 0$$
$$\delta(A, B) \leq \delta(A, C) + \delta(C, B)$$
$$\delta(A, B) = 0 \text{ iff } A = B = \{a\},$$
$$\delta(A, A) = \text{diam } A,$$

**Definition 1.1 [2]:** A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if

(i) given a $\in$ A, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for n = 1, 2, ..... , and $\{a_n\}$ converges to a

(ii) given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for n > N where $A_\varepsilon$ is the union of all open spheres with centers in A and radius $\varepsilon$.

**Definition 1.2 [2] :** A set valued mapping F of X into B(X) is said to be continuous at x $\in$ X if the sequence $\{Fx_n\}$ in B(X) converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X, F is said to be continuous on X if it is continuous at every point X.

Let I: X $\rightarrow$ X be self mapping and f : X $\rightarrow$ B(X) a set valued mapping. Sessa et al. [7] defines I and f to be weakly commuting if Ifx $\in$ B(X) and

$$\delta(Ifx, fIx) \leq \text{Max}\{d(Ix, fx), \text{diam } Ifx\}$$

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Jungck and Rhoades [3] defines I and f to be $\delta$ – compatible if

$$\lim_{n\to\infty} \delta(fIx_n, Ifx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \{t\} \text{ and } \qquad \lim_{n\to\infty} Ix_n = t$$

for some $t \in X$. Clearly, commuting mappings are weakly commuting and weakly commuting are $\delta$–compatible but neither implication is reversible as shown by example in [7] and [3] respectively.

Recently Jungck and Rhoades [3] defines I and f to be weakly compatible if for each point u in X such that fu = {Iu}, we have flu = Ifu. In [3], it is shown that $\delta$ – compatible mappings are weakly compatible but the converse need not to be true.

**IMPLICIT RELATIONS:** Let $F^*$ be the collection of real functions $F(t_1,....t_6)$: $(R_+)^6 \to R$ satisfying the following conditions:

($F_1$): f is non increasing in each co – ordinate variable except $t_1$,
($F_2$): $F(u, v, v, u, u + v, 0) < 0$ or $F(u, v, u, v, 0, u + v) < 0$ implies $u < v$.
($F_3$): $F(u, u, 0, 0, u, u) \geq 0$ for all $u > 0$.

**Example 1.1:** $F(t_1, ......, t_6) = t_1 - \max\{t_2, t_3, t_4\}$.

($F_1$): obviously
($F_2$): Let $F(u, v, v, u, u + v, 0) = u - \max\{v, v, u\} < 0$, if $u \geq v$ then $u < u$ a contradiction.

Thus $u < v$. Similarly if $F(u, v, u, v, 0, u + v) < 0$ then $u < v$.

($F_3$): $F(u, u, 0, 0, u, u) = 0$ for all $u > 0$.

**Example 1.2:** $F(t_1, ......, t_6) = t_1 - \max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}, \}$.
$F_1$, $F_2$ and $F_3$ can be shown as in example 1.1.

**Example 1.3:** $F(t_1, ......., t_6) = t_1 - \max\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\}$.
$F_1$, $F_2$ and $F_3$ can be shown as in example 1.1.

**Example 1.4:** $F(t_1, ...., t_6) = t_1^2 - c_1\max\{t_2^2, t_3^2, t_4^2\} - c_2\{t_3t_4, t_4t_6\} - c_3\{t_5t_6\}$, where $c_1 + 2c_2 \leq 1$,

$c_1 + c_3 \leq 1$ and $c_1, c_2, c_3 \geq 0$.

($F_1$): obviously
($F_2$): Let $F(u, v, v, u, u + v, 0) = u^2 - \max\{u^2, v^2\} < 0$, if $u \geq v$ then $u < u$, a contradiction.

Thus $u < v$. Similarly if $F(u, v, u, v, 0, u + v) < 0$ then $u < v$.

($F_3$): $F(u, u, 0, 0, u, u) = u^2\{1- (c_1 + c_3)\} \geq 0$, for all $u > 0$.

**Example 1.5:** $F(t_1, ........, t_6) = t_1 - \max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}, b\sqrt{(t_5 t_6)}\}$, where $0 < b < 1$
$F_1$ and $F_2$ can be shown as in example 1.1.

($F_3$): $F(u, u, 0, 0, u, u) = u - \text{Max}\{u, bu\} \geq 0$ for all $u > 0$.

**Example 1.6:** $F(t_1, ......, t_6) = t_1 - \alpha.\max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$,

where $0 \leq \alpha < 1$,

$$0 \leq a \leq \frac{1}{2} \quad \text{and} \quad 0 \leq b \leq \frac{1}{2}$$

($F_1$): obviously

($F_2$): Let $u > 0$, $v > 0$ and $F(u, v, v, u, u + v, 0) = u - \alpha.\max\{v, v, u\} - (1 - \alpha)a(u + v) < 0$.

If $u \geq v$, then $(1 - \alpha)(1 - 2a)u < 0$ a contradiction. Thus $u < v$.

Similarly $F(u, v, u, v, 0, u + v) < 0$ implies $u < v$ . If $u = 0$, $v > 0$ then $u < v$.

($F_3$): $F(u, u, 0, 0, u, u) = u(1 - \alpha)\ (1 - (a + b)) \geq 0$, for all $u > 0$.

**Example 1.7:** $F(t_1, \ldots\ldots, t_6) = t_1^3 - a\,t_1^2\,t_2 - bt_1t_3t_4 - c\,t_5^2\,t_6 - dt_5\,t_6^2$ where $a, b, c, d \geq 0$ and $a + b + c + d < 1$.

($F_1$): Obviously

($F_2$): Let $u > 0$, $v > 0$ and $F(u, v, v, u, u+v, 0) = u^2\,\{u - (a + b)v\} < 0$, which implies

$u < (a + b)\,v < v$. If $u = 0$, $v > 0$, then $u < v$.

Similarly $F(u, v, u, v, 0, u + v) < 0$ implies $u < v$.

($F_3$): $F(u, u, 0, 0, u, u) = u^3\,(1 - (a + b + c + d) \geq 0$ for all $u > 0$.

## 2. PRELIMINARIES

The following theorems are proved in [1], [2], [5], [7], [9] and [10]:

**Theorem 2.1 [2]:** Let F, G be continuous mappings of a compact metric space (X, d) into B(X) and I, J continuous mappings of X into itself satisfying the inequality

$$d(Fx, Gy) < \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\}, \tag{2.1}$$

for all $x, y \in X$ for which the righthand side of the inequality (2.1) is positive. If the mapping F and I commute and G and J commute and $G(X) \subset J(X)$, $F(X) \subset J(X)$, then there is a unique point u in X such that

$$Fu = Gu = \{u\} = \{Iu\} = \{Ju\}.$$

**Theorem 2.2 [1]:** Let I, J be functions of a compact metric space (X, d) into itself and $F, G: X \to B(X)$ two set–valued functions with $\cup F(X) \subseteq J(X)$ and $\cup G(X) \subseteq I(X)$. Suppose that the inequality

$$\delta(Fx, Gy) < \alpha.\max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\} + (1 - \alpha)\,[aD(Ix, Gy) + bD(Jy, Fx)], \tag{2.2}$$

for all $x, y \in X$, where $0 \leq \alpha < 1$, $a \geq 0$, $b \geq 0$, $a \leq \frac{1}{2}$, $b < \frac{1}{2}$, $\alpha\,|a - b| < 1 - (a + b)$, holds whenever the righthand side of (2.2) is positive. If the pairs {F, I} and {G, J} are weakly compatible, and if the functions F and I are continuous, then there is a unique point u in X such that

$$Fu = Gu = \{u\} = \{Iu\} = \{Ju\}.$$

**Theorem 2.3 [4] :** Let A, B, S, T be continuous self mappings of a compact metric space with $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If {A, S} and {B, T} are compatible pairs and

$$d(Ax, By) < \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \tfrac{1}{2}(d(Ax, Ty) + d(By, Sx))\} \tag{2.3}$$

for all x, y in X for which the right hand side of (7.4.4) is positive. Than A, B, S, T have a unique common fixed point.

**Theorem 2.4 [7]:** Let S and I be self mappings of a non empty compact metric space (X, d) satisfying

$$d(Sx, Ty) < \max\{d(x, y), \tfrac{1}{2}(d(x, Sx) + d(y, Ty)), \tfrac{1}{2}(d(x, Ty) + d(y, Tx))\} \tag{2.4}$$

If S or T is continuous then S and T has a unique common fixed point.

 

**Theorem 2.5 [9] :** Let A, B, S and T be continuous self maps of a compact metric space (X, d) with $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If {A, S} and {B, T} are compatible pairs and

$$d^2(Ax, By) < c.\max\{d^2(Sx, Ax), d^2(Ty, By), d^2(Sx, Ty)\} + \frac{1}{2}(1-c).\max\{d(Sx, Ax)\, d(Sx, By), d(Ax, Ty), d(By, Ty)\} + (1-c)\, d(Sx, By).d(Ty, Ax) \qquad (2.5)$$

for all x, y in X for which the right hand side of (2.5) is positive, where $c \in (0, 1)$. Then A, B, S and T have a common fixed point z. Further, z is the unique common fixed point of A and S and of B and T.

**Theorem 2.6 [10]:** Let S and T be continuous self mappings of a compact metric space (X, d) satisfying inequality

$$d(Sx, Ty) < \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}\{d(x, Ty) + d(y, Sx)\}, b\sqrt{d(x, Ty).d(y, Sx)}\} \qquad (2.6)$$

for all x, y in X for which the right hand side of (2.6) is positive, where $b > 0$. Then S and T have a common fixed point. Further, if $b < 1$, then the common fixed point is unique.

## 3. MAIN RESULT

We prove the following theorem

**Theorem 3.1:** Let I, J be self mappings of a compact metric space (X, d) and f, g: $X \to B(X)$ two set valued mappings satisfying

(i) $\cup f(X) \subset J(X)$ and $\cup g(X) \subset I(X)$,
(ii) $F\{\delta(fx, gy), d(Ix, Jy), \delta(Ix, fx), \delta(Jy, gy), D(Ix, gy), D(Jy, fx)\} < 0$
for all x, y in X for which at least one of $d(Ix, Jy)$, $\delta(Ix, fx)$, $\delta(Jy, gy)$ is positive, where $F \in F^*$.
(iii) The pair {f, I} and {g, J} are weakly compatible,
(iv) The mapping f and I are continuous.

Then there exists a unique point $u \in X$ such that $fu = gu = \{u\} = \{Iu\} = \{Ju\}$.

**Proof:** Let $\varepsilon = \mathrm{Inf}\,\{\delta(Ix, fx): x \in X\}$. Since X is compact space, there is a convergent sequence $\{x_n\}$ with limit $x_0$ in X such that

$$\lim_{n\to\infty} \delta(Ix_n, fx_n) = \varepsilon.$$

Since $\delta(Ix_0, fx_0) \leq d(Ix_0, Ix_n) + \delta(Ix_n, fx_n) + \delta(fx_n, fx_0)$,

therefore by the continuity of f and I and $\lim_{n\to\infty} x_n = x_0$, we get $\delta(Ix_0, fx_0) \leq \varepsilon$ and that $\delta(Ix_0, fx_0) = \varepsilon$.

Since $\cup f(X) \subseteq J(X)$, there exists a point $y_0 \in X$ such that $Jy_0 \in fx_0$ and $d(Ix_0, Jy_0) \leq \varepsilon$.

If $\varepsilon > 0$, then, by (ii) we have

$$F\{\delta(fx_0, gy_0), \delta(Ix_0, Jy_0), \delta(Ix_0, fx_0), \delta(Jy_0, gy_0), D(Ix_0, gy_0), D(Jy_0, fx_0)\} < 0$$

$$\Rightarrow \quad F\{\delta(fx_0, gy_0), \varepsilon, \varepsilon, \delta(fx_0, gy_0), \delta(fx_0, gy_0) + \varepsilon, 0\} < 0.$$

By $(F_2)$ it implies

$$\delta(fx_0, gy_0) < \varepsilon \text{ and hence } \delta(Jy_0, gy_0) \leq \delta(fx_0, gy_0) < \varepsilon.$$

Since $\cup g(X) \subset I(X)$, then there exists a point $z_0$ in X such that $Iz_0 \in gy_0$ and $d(Iz_0, Jy_0) < \varepsilon$.

Now, since $\delta(Iz_0, Jy_0) \geq \varepsilon > 0$. Then, we have,

$$F\{\delta(fz_0, gy_0), \delta(Iz_0, Jy_0), \delta(Iz_0, fz_0), \delta(Jy_0, gy_0), D(Iz_0, gy_0), D(Jy_0, fz_0)\} < 0$$

$$F\{\delta(fz_0, gy_0), \delta(Jy_0, gy_0), \delta(fz_0, gy_0), \delta(Jy_0, gy_0), 0, \delta(fz_0, gy_0) + \delta(Jy_0, gy_0)\} < 0$$

 

which by ($F_2$) yields $\delta(fz_0, gy_0) < \delta(Jy_0, gy_0)$, but then,

$$\varepsilon \le \delta(Iz_0, fz_0) \le \delta(fz_0, gy_0) < \delta(Jy_0, gy_0) < \varepsilon.$$

a contradiction. Thus $\varepsilon = 0$. Then we get $\{Ix_0\} = \{Jy_0\} = fx_0$.

If $\delta(Jy_0, gy_0) > 0$ then by (ii), we have

$$F\{\delta(fx_0, gy_0), d(Ix_0, Jy_0), \delta(Ix_0, fx_0), \delta(Jy_0, gy_0), D(Ix_0, gy_0), D(Jy_0, fx_0)\} < 0$$

$$F\{\delta(Jy_0, gy_0), 0, 0, \delta(Jy_0, gy_0), \delta(Jy_0, gy_0), 0\} < 0$$

which, by ($F_2$), implies that $\delta(Jy_0, gy_0) < 0$, a contradiction. Thus $\delta(Jy_0, gy_0) = 0$ and so $gy_0 = \{Jy_0\}$.

Therefore $\{Ix_0\} = fx_0 = \{Jy_0\} = gy_0 = \{p\}$, (say) (3.1)

Then, by weak compatibility of the pair {f, I} we have

$$fp = f(Ix_0) = \{Ifx_0\} = \{Ip\} \tag{3.2}$$

If $Ip \ne p = Jy_0$, then by an application of (ii), we have,

$$F\{\delta(fp, gy_0), d(Ip, Jy_0), \delta(Ip, fp), \delta(Jy_0, gy_0), D(Ip, gy_0), D(Jy_0, fp)\} < 0, \tag{3.3}$$

Now using, (3.1), (3.2) and (3.3), we get

$$F\{d(fp, p), d(fp, p), 0, 0, d(fp, p), d(fp, p)\} < 0$$

which, by ($F_3$), is a contradiction. Therefore $d(fp, p) = 0$ and hence $fp = \{p\}$ and so

$$fp = \{Ip\} = \{p\}. \tag{3.4}$$

Now, since J and g are weakly compatible $\{Jp\} = \{Jgy_0\} = gJy_0 = gp$. Suppose $Ip \ne Jp$, then $d(Ip, Jp) > 0$ and so

$$F\{\delta(fp, gp), d(Ip, Jp), \delta(Ip, fp), \delta(Jp, gp), D(Ip, gp), D(Jp, fp)\} < 0$$

$$F\{d(Ip, Jp), d(Ip, Jp), 0, 0, d(Ip, Jp), d(Ip, Jp)\} < 0$$

which by {$F_3$}, is a contradiction. Thus $Ip = Jp$ and hence

$$fp = gp = \{Ip\} = \{Jp\} = \{p\}.$$

Again suppose, q be a point such that, i.e. $fq = gq = \{Iq\} = \{Jq\} = q$. Then, by (ii) we have,

$$F\{\delta(fp, gq), d(Ip, Jq), \delta(Ip, fp), \delta(Jq, gq), D(Ip, gq), D(Jq, fp)\} < 0$$

$$F\{d(p, q), d(p, q), 0, 0, d(p, q), d(p, q)\} < 0$$

which by ($F_3$), yields $d(p, q) = 0$ and so $p = q$.

**Corollary 3.1:** Let
(i), J: $X \to X$ be self mapping of compact metric spaces (X, d) and f, g set valued mappings satisfying (i), (iii), (iv) and (ii) $\delta(fx, gy) < \max\{d(Ix, Jy), \delta(Ix, fx), \delta(Jy, gy)\}$

for all $x, y \in X$ for which the right hand side of the inequality (ii)* is positive. Then f, g, I and J have a unique common fixed point.

**Proof:** Follows from theorem 3.1 and example 1.2.

**Remark 3.1:** The corollary 3.1 generalizes theorem 2.1.

**Corollary 3.2:** Theorem 2.2.

**Proof:** Follows from theorem 3.1 and example 1.6.

**Remark 3.2:** Theorem 2.2 holds, only for $0 \leq a \leq \frac{1}{2}$ and $0 \leq b < \frac{1}{2}$ while the result obtained by theorem 3.1 and example 1.6. [Corollary 3.2] holds for $0 \leq a \leq \frac{1}{2}$ and $0 \leq b \leq \frac{1}{2}$. Thus theorem 3.1 is a proper generalization of theorem 2.2.

**Remark 3.3:** Theorem 3.1 and example 1.2 yields the extension of theorem 2.3 for self mappings, to set valued mappings with more weakened condition of weak compatibility.

(B). Theorem 3.1 with I = J = identity mapping and example 1.3 provide the extension of theorem 2.4 for set valued mappings.

(C) Theorem 3.1 and example 1.4 with $c_1= c$, $c_2= \frac{1}{2}(1–c)$ and $c_3 = (1 – c)$ gives the extension of theorem 2.5.

(D). Theorem 3.1 and example 1.5 with I = J identity mapping gives the extension of theorem 2.6.

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