International Journal of Mathematical Archive-3(6), 2012, 2445-2450
Available online through www.ijma.info ISSN 2229 – 5046

COMMON FIXED POINTS OF A-COMPATIBLE AND S-COMPATIBLE MAPPINGS

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(Received on: 30-05-12; Revised & Accepted on: 19-06-12)

ABSTRACT

In this paper we prove two common fixed point theorems of A-compatible and S-compatible mappings. Our results modify results of [1, 4, 5, 6, 7].

Keywords: *Fixed point, complete metric space, compatible mappings, A-compatible, S-compatible.*

2000 AMS Classification*: 47H10, 54H25.*

1. INTRODUCTION

The first important result in the theory of fixed point of compatible mappings was obtained by Gerald Jungck in 1986 [2] as a generalization of commuting mappings. In 1993 Jungck, Murthy and Cho [3] introduced the concept of compatible mappings of type (A) by generalizing the definition of weakly uniformly contraction maps. Pathak and Khan [5] introduced the concept of A-compatible and S-compatible by splitting the definition of compatible mappings of type (A).

The aim of this paper is to prove two common fixed point theorems by using the concept of A-compatible and S-compatible mappings in metric spaces considering four self mappings.

Following are definitions of types of compatible mappings.

Definition 1.1 [2]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 1.2 [3]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be compatible of type (A) if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that for $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 1.3 [5]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be A-compatible if $\lim_{n\to\infty} d\,(ASx_n, SSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that for $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 1.4 [5]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be S-compatible if $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that for $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Proposition 1.5[6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is A-compatible on X and $St = At$ for $t \in X$, then $ASt = SSt$.

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Proof: Let $\{x_n\}$ be a sequence in X defined by $x_n=t$ for $n=1, 2\ldots$ and let $At=St$. Then we have $\lim_{n\to\infty} Ax_n = At$ and $\lim_{n\to\infty} Sx_n = St$. Since the pair (A, S) is A-compatible we have

$$d(ASt, SSt)= \lim_{n\to\infty} d(ASx_n, SSx_n) = 0.$$

Hence $ASt=SSt$.

Proposition 1.6[6]**:** Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is S-compatible on X and $St = At$ for $t \in X$, then $SAt = AAt$.

Proof: Let $\{x_n\}$ be a sequence in X defined by $x_n=t$ for $n=1, 2, \ldots.$ and let $At=St$. Then we have $\lim_{n\to\infty} Ax_n = At$ and $\lim_{n\to\infty} Sx_n = St$. Since the pair (A, S) is S-compatible we have

$$d(SAt, AAt)= \lim_{n\to\infty} d(SAx_n, AAx_n) = 0.$$

Hence $SAt=AAt$.

Proposition 1.7[6]**:** Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is A-compatible on X and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for $t \in X$, then $SSx_n \to At$ if A is continuous at t.

Proof: Since A is continuous at t we have $ASx_n \to At$. Since the pair (A, S) is A-compatible, we have $d(ASx_n, SSx_n) = 0$ as $n \to \infty$. It follows that

$$d(At, SSx_n) \leq d(At, ASx_n) + d(ASx_n, SSx_n)$$

Therefore, $\lim_{n\to\infty} d(At, SSx_n) = 0$.

And so we have $SSx_n \to At$ as $n\to\infty$.

Proposition 1.8[6]**:** Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is S-compatible on X and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for $t \in X$, then $AAx_n \to St$ if S is continuous at t.

Proof: Since S is continuous at t we have $SAx_n \to St$. Since the pair (A, S) is S-compatible, we have $d(SAx_n, AAx_n) = 0$ as $n \to \infty$. It follows that

$$d(St, AAx_n) \leq d(St, SAx_n) + d(SAx_n, AAx_n)$$

Therefore, $\lim_{n\to\infty} d(St, AAx_n) = 0$.

And so we have $AAx_n \to St$ as $n\to\infty$.

2. MAIN RESULTS

We need the following lemma.

Lemma 2.1[1]**:** Let A, B, S and T be mapping from a metric space (X, d) into itself satisfying the following conditions:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2) $[d(Ax, By)]^2 \leq a[d(Ax, Sx)d(By, Ty)+d(By, Sx)d(Ax, Ty)] + b[d(Ax, Sx)d(Ax, Ty)+d(By, Ty)d(By, Sx)]$

where $0 \leq a + 2b < 1$; $a, b \geq 0$

(3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1}=Tx_{2n+1}=Ax_{2n} \text{ and } y_{2n}= Sx_{2n}= Bx_{2n-1}$$

then the sequence $\{y_n\}$ is Cauchy sequence in X.

Proof: By condition (2) and (3), we have

$$[d(y_{2n+1}, y_{2n})]^2=[d(Ax_{2n}, Bx_{2n-1})]^2$$

$$\leq a[d(Ax_{2n}, Sx_{2n})d(Bx_{2n-1}, Tx_{2n-1})+d(Bx_{2n-1}, Sx_{2n})d(Ax_{2n}, Tx_{2n-1})] + b[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n-1})+d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, Sx_{2n})]$$

$$= ad(y_{2n+1}, y_{2n})d(y_{2n}, y_{2n-1}) + bd(y_{2n+1}, y_{2n})d(y_{2n+1}, y_{2n-1})$$

$$[d(y_{2n+1}, y_{2n})] \leq ad(y_{2n}, y_{2n-1}) + b[d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})]$$

$$[d(y_{2n+1}, y_{2n})] \leq pd(y_{2n}, y_{2n-1}) \text{ where } p = \frac{a+b}{1-b} < 1.$$

Hence $\{y_n\}$ is Cauchy sequence.

Now we give our main theorem.

Theorem 2.2: Let A, B, S and T be self maps of a complete metric space (X, d) satisfying the following conditions:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2) $[d(Ax, By)]^2 \leq a[d(Ax, Sx)d(By, Ty)+d(By, Sx)d(Ax, Ty)] + b[d(Ax, Sx)d(Ax, Ty)+d(By, Ty)d(By, Sx)]$

where $0 \leq a + 2b< 1$; $a, b\geq 0$

(3) Let $x_0 \in X$ then by (1) there exists $x_1\in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2\in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1}=Tx_{2n+1}=Ax_{2n} \text{ and } y_{2n}= Sx_{2n}= Bx_{2n-1}$$

then the sequence $\{y_n\}$ is Cauchy sequence in X.

(4) One of A, B, S or T is continuous.

(5) (A, S) and (B, T) are A-compatible on X.

Then A, B, S and T have a unique common fixed point in X.

Proof: By lemma 2.1, $\{y_n\}$ is Cauchy sequence and since X is complete so there exists a point $z\in X$ such that $\lim y_n = z$ as $n \to \infty$. Consequently subsequences Ax_{2n}, Sx_{2n}, Bx_{2n-1} and Tx_{2n+1} converges to z.

Let A be continuous. Since A and S are A-compatible on X, then by proposition 1.7 we have $S^2x_{2n} \to Az$ and $ASx_{2n} \to Az$ as $n \to \infty$.

Now by condition (2) of lemma 2.1, we have

$$[d(ASx_{2n}, Bx_{2n-1})]^2 \leq a[d(ASx_{2n}, S^2x_{2n})d(Bx_{2n-1}, Tx_{2n-1})+d(Bx_{2n-1}, S^2x_{2n})d(ASx_{2n-1}, Tx_{2n-1})] + b\,[d(ASx_{2n}, S^2x_{2n})d(ASx_{2n}, Tx_{2n-1}) +d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, S^2x_{2n})]$$

As $n\to\infty$, we have

$$[d(Az, z)]^2\leq a[d(Az, z)]^2,$$

which is a contradiction. Hence $Az = z$,

Now since $Az = z$, by condition (1) $z \in T(X)$. Also T is self map of X so there exists a point $u \in X$ such that $z = Az = Tu$. More over by condition (2), we obtain,

$$[d(z, Bu)]^2=[d(Az, Bu)]^2\leq a[d(Az, Sz)d(Bu, Tu) + d(Bu, Sz)d(Az, Tu)] + b[d(Az, Sz)d(Az, Tu) + d(Bu, Tu)d(Bu, Sz)]$$

i.e., $[d(z, Bu)]^2 \leq b\,[d(z, Bu)]^2$.

Hence $Bu = z$ i.e., $z = Tu = Bu$.

By proposition 1.7, we have $BTu=TTu$

Hence $Bz = Tz$.

Now,

$$[d(z, Tz)]^2=[d(Az, Bz)]^2\le a[d(Az, Sz)d(Bz, Tz) + d(Bz, Sz)d(Az, Tz)] + b[d(Az, Sz)d(Az, Tz) + d(Bz, Tz)d(Bz, Sz)]$$

i.e., $[d(z, Tz)]^2 \le a[d(z, Tz)]^2$ which is a contradiction. Hence $z = Tz$ i.e, $z = Tz = Bz$.

Now since $Bz = z$, by condition (1) $z\in S(X)$. Also S is self map of X so there exists a point $v\in X$ such that $z = Bz = Su$. Moreover by (2) we have

$$[d(Au, z)]^2=[d(Au, Bz)]^2\le a[d(Au, Su)d(Bz, Tz) + d(Bz, Su)d(Au, Tz)] + b[d(Au, Su)d(Au, Tz) + d(Bz, Tz)d(Bz, Su)]$$

i.e., $[d(Au, z)]^2 \le b[d(Au, z)]^2$.

Hence $Au = z$ i.e., $z = Au = Su$.

By proposition 1.5, we have $ASu=SSu$

Hence $Az = Sz$.

Therefore z is common fixed point of A, B, S and T. Similarly we can prove this when any one of A, B or T is continuous.

Finally, in order to prove the uniqueness of z, suppose w be another common fixed point of A, B, S and T then we have,

$$[d(z, w)]^2=[d(Az, Bw)]^2\le a[d(Az, Sz)d(Bw, Tw) + d(Bw, Sz)d(Az, Tw)] + b[d(Az, Sz)d(Az, Tw) + d(Bw, Tw)d(Bw, Sz)]$$

which gives

$$[d(z, Tw)]^2 \le a[d(z, Tw)]^2. \text{ Hence } z = w.$$

This completes the proof.

Theorem 2.3: Let A, B, S and T be self maps of a complete metric space (X, d) satisfying the following conditions:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2) $[d(Ax, By)]^2 \le a[d(Ax, Sx)d(By, Ty)+d(By, Sx)d(Ax, Ty)] + b[d(Ax, Sx)d(Ax, Ty)+d(By, Ty)d(By, Sx)]$

where $0 \le a + 2b< 1$; $a, b\ge 0$

(3) Let $x_0 \in X$ then by (1) there exists $x_1\in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2\in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1}=Tx_{2n+1}=Ax_{2n} \text{ and } y_{2n}= Sx_{2n}= Bx_{2n-1}$$

then the sequence $\{y_n\}$ is Cauchy sequence in X.

(4) One of A, B, S or T is continuous.

(5) (A, S) and (B, T) are S-compatible on X.

Then A, B, S and T have a unique common fixed point in X.

Proof: By lemma 2.1, $\{y_n\}$ is Cauchy sequence and since X is complete so there exists a point $z\in X$ such that $\lim y_n = z$ as $n \to \infty$. Consequently subsequences Ax_{2n}, Sx_{2n}, Bx_{2n-1} and Tx_{2n+1} converges to z.

Let S be continuous. Since A and S are S-compatible on X, then by proposition 1.8 we have $SAx_{2n} \to Sz$ and $AAx_{2n} \to Sz$ as $n \to \infty$.

by condition (2) of lemma 2.1, we have

$$[d(AAx_{2n}, Bx_{2n-1})]^2 \leq a[d(AAx_{2n}, SAx_{2n})d(Bx_{2n-1}, Tx_{2n-1}) + d(Bx_{2n-1}, SAx_{2n})d(AAx_{2n-1}, Tx_{2n-1})] + b[d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n-1}) + d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, SAx_{2n})]$$

As $n\to\infty$, we have

$$[d(Sz, z)]^2 \leq a[d(Sz, z)]^2,$$

which is a contradiction. Hence $Sz = z$,

Now $$[d(Az, Bx_{2n-1})]^2 \leq a[d(Az, Sz)d(Bx_{2n-1}, Tx_{2n-1}) + d(Bx_{2n-1}, Sz)d(Az, Tx_{2n-1})] + b[d(Az, Sz)d(Az, Tx_{2n-1}) + d(Bx_{2n-1}, Tx_{2n-1})d(Bx_{2n-1}, Sz)]$$

Letting $n\to\infty$, we have $[d(Az, z)]^2 \leq b[d(Az, z)]^2$. Hence $Az = z$.

Now since $Az = z$, by condition (1) $z \in T(X)$. Also T is self map of X so there exists a point $u \in X$ such that $z = Az = Tu$. More over by condition (2), we obtain,

$$[d(z, Bu)]^2=[d(Az, Bu)]^2 \leq a[d(Az, Sz)d(Bu, Tu) + d(Bu, Sz)d(Az, Tu)] + b[d(Az, Sz)d(Az, Tu) + d(Bu, Tu)d(Bu, Sz)]$$

i.e., $[d(z, Bu)]^2 \leq b[d(z, Bu)]^2$.

Hence $Bu = z$ i.e., $z = Tu = Bu$.

By proposition 1.6, we have $TBu = BBu$

Hence $Tz = Bz$.

Now,

$$[d(z, Tz)]^2=[d(Az, Bz)]^2 \leq a[d(Az, Sz)d(Bz, Tz) + d(Bz, Sz)d(Az, Tz)] + b[d(Az, Sz)d(Az, Tz) + d(Bz, Tz)d(Bz, Sz)]$$

i.e., $[d(z, Tz)]^2 \leq a[d(z, Tz)]^2$ which is a contradiction. Hence $z = Tz$ i.e, $z = Tz = Bz$.

Therefore z is common fixed point of A, B, S and T. Similarly we can prove this when any one of A, B or T is continuous.

Finally, in order to prove the uniqueness of z, suppose w be another common fixed point of A, B, S and T then we have,

$$[d(z, w)]^2=[d(Az, Bw)]^2$$

$$\leq a[d(Az, Sz)d(Bw, Tw) + d(Bw, Sz)d(Az, Tw)] + b[d(Az, Sz)d(Az, Tw) + d(Bw, Tw)d(Bw, Sz)]$$

which gives

$$[d(z, Tw)]^2 \leq a[d(z, Tw)]^2.$$ Hence $z = w$.

This completes the proof.

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Source of support: Nil, Conflict of interest: None Declared