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SOME RESULTS OF FIXED POINTS ON GROUPS

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ABSTRACT

In 2006 J. Achari & Neeraj Anant Pande [1] established fixed point theorems for a family of self maps on groups using the following concept.

Let $(G.*)$ *be a group and* $f_i : G \to G$ *be a self map on G given by* $f_i(g) = g^i$ *for each* $g \in G$ *then* $x \in G$ *is a fixed point of* f_i *iff* $O(x)|i-1$.

In this paper we established some results of fixed points on groups by using the above concept.

***Key words**: Groups, sub groups, normal sub groups, Homomorphism.*

***AMS Subject Classification**: 47H10, 54H25.*

1. INTRODUCTION

A fixed point is a point which remains in variant under a mapping from a set X to a set Y with $X \cap Y \neq \phi$. It is an intersting fact that finding of solution for the equation $x^2 - 3x + 2 = 0$ is same as finding fixed points for the mapping $f(x) = \frac{x^2+2}{3}$. Clearly 1, 2 are fixed points of f and are the solutions of the above equation.

In 2006 J. Achari & Neeraj Anant Pande [1] established fixed point theorems for a family of self maps on groups.

They established some results by using the following concept [1] . Let $(G.*)$ be a group. Consider a family of self maps.

$\{f_i : G \to G / i \in I\}$ on G given by

$f_i(g) = g^i$ for each $g \in G$. Let F_{f_i} denote the set of all fixed points of map f_i.

Theorem 1[1]: Let $(G.*)$ be a group and $f_i : G \to G$ be a self map on G given by $f_i(g) = g^i$ for each $g \in G$. Then $x \in G$ is a fixed point of f_i iff $O(x)|i-1$.

Theorem 1[1] immediately gives that in an infinite group, an element is of infinite order then it can not be a fixed point of any of the maps f_i's for $i \neq 1$.

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The following example supports the statement.

Example 1[1]: The set **I** of all integers is an abelian group with respect to the usual addition of integers. In this group every non identity element is of infinite order and can not be fixed point for any of the maps.

$$f_i : I \to I,\ i \in I,\ (i \neq 1)$$, given by

$$f_i(n) = n + n + ... + n(i\ \text{times}) = in \text{ for } i \geq 0$$

and $$f_i(n) = (-n) + (-n) + ... + (-n)(-i\ \text{times}) = in, \text{ for } i < 0$$

Example 2[1]: Let $(G.*)$ be a group and for each $i \in I$, let the self map $f_i : G \to G$ on G given by $f_i : (g) = g^i$ for each $g \in G$. Then each f_i has at least one fixed point. viz, the identity element e,

thus $e \in F_{f_i} \neq \phi$ for each i.

Theorem 3[1]: For a group $(G.*)$, consider a self map $f_i : G \to G$ on G given by $f_i(g) = g^i$ for each $g \in G$. Then $x \in G$ is a fixed point f_i of iff x^{-1} is a fixed point.

Theorem 4[1]: For a group $(G.*)$ suppose a self map $f_i : G \to G$ on G given by $f_i(g) = g^i$ for each $g \in G$ is a homomorphism. The x and y are fixed points of implies that x*y is also fixed point of f_i. In this case F_{f_i}, itself is group w.r.t*, and hence a subgroup of G.

If G is abelian group, then each f_i is a homomorphism and F_{f_i} is a subgroup of G. Moreover it is abelian.

We established the following.

Theorem 1: For an abelian group $(G,*)$ consider a self map $f_i : G \to G$ on G given by $f_i(g) = g^i$ for each of $g \in G$, F_{f_i} is set of all fixed points of $\mathbf{f_i}$ then $\mathbf{F_{f_i}}$ is a normal subgroup of G.

Proof: Clearly from Theorem 4[1] F_{f_i} is a subgroup of G.

Let $x \in G,\ y \in F_{f_i}$

$\Rightarrow x \in G,\ f_i(y) = y = y^i$

Since G is abelian f_i is homomorphism

$$f_i(xyx^{-1}) = f_i(x) f_i(y) f_i(x^{-1}) = x^i y.(x^{-1})^i = y\left(x^i (x^{-1})^i\right) = ye = yxx^{-1} = xyx^{-1}$$

$\Rightarrow \mathbf{xyx^{-1}}$ is a fixed point of $\mathbf{f_i}$.

$\therefore xyx^{-1} \in F_{f_i} .. F_{f_i}$ is a normal subgroup of G.

Theorem 2: For any group $(G,*)$ consider any self map $f_i : G \to G$ on G given by $f_i(g) = g^i$ for each of $g \in G$, then F_{f_i} (set of all fixed points of f_i) and ker f_i are such that

ker f_i is a subgroup of F_{f_i} iff ker $f_i = \{e\}$

Proof: I, Suppose ker f_i is a subgroup of F_{f_i}

claim: ker $f_i = \{e\}$

ker $f_i \subseteq F_{f_i}$

Let $x \in$ ker $f_i \Rightarrow x \in F_{f_i}$

Since $x \in$ ker f_i, $f_i(x) = e$ (1)

Since $x \in F_{f_i}, f_i(x) = x^i = x$ (2)

From (1) and (2) $x = e$.. this is true for every $x \in$ ker f_i

$\therefore \ker f_i = \{e\}$

II. Suppose ker $f_i = \{e\}$

claim: ker f_i is a subgroup of F_{f_i}

$e \in \ker f_i,\ f_i(e) = e^i = e \Rightarrow F_{f_i}$

$\Rightarrow \ker f_i \subseteq F_{f_i}$

To show ker f_i is a subgroup of $F_{f_i} x = e,\ y = e \in$ ker f_i

$\Rightarrow xy^{-1} = e \in$ ker f_i.

$\therefore$ ker f_i is a subgroup of F_{f_i}.

Theorem 3: Let $(G,*)$ be an abelian group. $f_i : G \to G$ on G given by $f_i(g) = g^i$ for each of $g \in G$. If a,b are fixed points of f_i and then a b is a fixed point of f_i. Converse is also true with the extra condition $(O(a), O(b)) = 1$.

Proof: To prove the theorem we use the following lemma.

Lemma: G is an abelian group. If $a,b \in G$ such that $O(a) = m$, $O(b) = n$ and $(m,n) = 1$ then $O(ab) = mn$.

Proof: G is an abelian group. Let e be indentity in G. Since $a,b \in G$ such that $O(a) = m,\ O(b) = n$, we have $a^m = e$ and $b^n = e$ also $ab \in G$. Let $O(ab) = p$

Now
$$\begin{aligned}(ab)^{mn} &= a^{mn}b^{mn} \\ &= \left((a)^m\right)^n \left((b)^n\right)^m \\ &= e^n e^m \\ &= e\end{aligned}$$

$\Rightarrow O(ab) | mn$

$\Rightarrow p | mn$ (1)

Also $(ab)^{pn} = \left((ab)^p\right)^n = e^n = e$

Again $(ab)^{pn} = a^{pn}b^{pn} = a^{pn}e^p = a^{pn}$

$a^{pn} = e \Rightarrow O(a) | pn \Rightarrow m | pn$

Since $(m,n) = 1,\ m | pn \Rightarrow m | p$ (2)

Similarly $n | p$ (3)

From (2), (3) and from $(m,n)=1$ we have mn/p (4)

From (1) and (4) $p=mn$

$O(ab)=O(a)O(b)$

Proof of main theorem:

I. Suppose a, b are fixed points of f_i.

Claim: ab is a fixed point of f_i.

Since a is a fixed point of f_i, $f_i(a)=a^i=a$ (5)

Again since b is fixed point of f_i, $f_i(b)=b^i=b$ (6)

$$f_i(ab)=(ab)^i$$
$$=a^ib^i$$
$$=ab \text{ from (5) and (6)}$$

$\Rightarrow ab$ is a fixed point of f_i

II. Supposed $(O(a),O(b))=1$ and ab is a fixed point of f_i.

Claim: a and b are fixed points of f_i.

Since $(O(a),O(b))=1$ by the above lemma

$$O(ab)=O(a)O(b)$$

Since ab is a fixed point of f_i, $O(ab)|i-1$.

$$\Rightarrow O(a)O(b)|i-1 \quad (7)$$

$$O(a)|O(a)O(b). \quad (8)$$

From (7) and (8) $O(a)|i-i \Rightarrow a$ is a fixed point of f_i.

$O(b)|O(a)\,O(b)$ (9)

i and -i are not fixed points of f_2 since $(O(i),\, O(-i))=4\neq 1$

Theorem 4: If x is a fixed point of f_i and f_j then x is alos a fixed point of $f_i \circ f_j$,

Proof: Since x is a fixed point of f_i, $O(x)|i-1$ again since x is a fixed point of f_j

$O(x)|j-1 \Rightarrow O(x)|(i-1)(j-1)$

$\Rightarrow O(x)|ij-i-j+1$

$O(x)|i-1,\, O(x)|ij-i-j+1$

$\Rightarrow O(x)|ij-j$

$O(x)|ij-j,\, O(x)|j-1 \Rightarrow O(x)|ij-1$

$\Rightarrow$ x is a fixed point of f_{ij}

$\Rightarrow$ x is a fixed point of $f_i \circ f_j$

But the converse is not true in view of the following example.

Example: $G=\{1,\omega,\omega^2\}$ is an abelian group.

$f_i : G \to G$ is defined as $f_i(x) = x^i$ for every $i \in Z$.

$$\left(f_5 \, 0 \, f_2\right)(\omega) = f_5\left(f_2(\omega)\right) = f_5\left(\omega^2\right) = \left(\omega^2\right)^5 = \omega^{10} = \omega$$

$\Rightarrow \omega$ is a fixed point of f_5 o f_2.

$$f_5(\omega) = \omega^5 = \omega^2 \neq \omega$$

$$f_2(\omega) == \omega^2 \neq \omega$$

$\Rightarrow \omega$ is not a fixed point of f_5 and f_2.

From (7) and (9) $O(b) | i-1 \Rightarrow b$ is a fixed point of f_i

Hence the converse is true. The following are the examples for justification of the statement.

Example 1: $G = \{1, -1, i, -i\}$ is an abelian group with respect to multiplication.

$f_i : G \to G$ is defined as $f_i(x) = x^i$ for every $i \in Z$.

i and -i are fixed points of f_5.

$$f_5(i) = i^5 = -i$$

$$f_5(-i) = (-i)^5 = -i$$

and (i) (-i)=1 is also a fixed point of f_5.

Example 2: $G = \{1, -1, i, -i\}$ is an abelina group with respect to multiplication.

$f_2 : G \to G$ is defined as $f_2(x) = x^2$ for every $x \in G$.

$$f_2(1) = 1^2 = 1$$

$\Rightarrow 1$ is a fixed point of f_2

but $(i)(-i) = 1$

$$f_2(i) = i^2 = -1 \neq i$$

$$f_2(-i) = (-i)^2 = -1 \neq -i$$

$\Rightarrow$ i and -i are not fixed points of f_2.

$$i^4 = 1 \Rightarrow O(i) = 4$$

$$(-i)^4 = 1 \Rightarrow O(-i) = 4$$

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