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Determination of the Shortest Path in Interval Valued Fuzzy Networks

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ABSTRACT

We propose a new approach to determine the shortest path in an Interval valued fuzzy networks (IVFN), a network in which vertices (or nodes) and edges (or links) remain crisp but each edge (i, i+1) has an associated weight, which is an interval fuzzy number of the form $R_i = [R_{iL}, R_{iU}]$ for each i. For each IVFN, we associate two fuzzy networks called lower and upper limit fuzzy networks having the same set of vertices and edges but each edge(i, i+1) is attached with a fuzzy weight R_{iL} and R_{iU} respectively. We exhibit that the shortest path of weight $w = [w_L, w_U]$ an interval fuzzy number in IVFN, is that path for which the shortest path of weight w_L in the lower limit fuzzy network coincides with the shortest path of weight w_U in the upper limit fuzzy network. The concept is illustrated with the help of a simple situation and the validation of mathematical verification is provided.

Keywords: *Fuzzy network, Interval valued fuzzy network, shortest path, Interval valued shortest path.*

MSC Subject Classification: *15B15, 15A09, 90C39, 94C15.*

1. INTRODUCTION

In graph theory the shortest path problem is the problem of finding a path between two vertices (or nodes) such that sum of the weight of its constituent edges is minimized. An example is finding the quickest way to get from one location to another on a road map; in this case, the vertices represent locations and are weighted by the time needed to travel that segment. The shortest path problem has transportation, communication routing and scheduling. Now, in any network path the arc length may represent time or cost. Therefore in real world, it can be considered to be a fuzzy set. Fuzzy set theory, proposed by Zadeh [10], is frequently utilized to deal with the uncertainty problem. We consider a directed network consisting of a finite set of vertices and a finite set of directed edges. It is assumed that there is only one directed edge between any two vertices. The fuzzy shortest path problem was first analyzed by Dubois and Prade [3]. They used Floyd's algorithm and Ford's algorithm to treat the fuzzy shortest path problem. Although in their method the shortest method length can be obtained, may be the corresponding path in the network doesn't exist. Klein [4], proposed a dynamical programming recursion – based fuzzy algorithm and later developed by many researchers [1, 2, 5, 6, 7]. Recently the concept of Interval valued fuzzy matrices (IVFM) as a generalization of fuzzy matrix was introduced and developed by Shyamal and Pal [9], by extending the max. min operations on Fuzzy algebra $\mathcal{F}=[0,1]$, for elements a, b$\in \mathcal{F}$, a + b = max {a, b} and a . b = min {a, b}. Let $\mathcal{F}_{mn}$ be the set of all mxn Fuzzy Matrices over the Fuzzy algebra with support [0, 1], that is matrices whose entries are intervals and all the intervals are subintervals of the interval [0, 1], then max $\{a_i, b_i\} = [\max\{a_{iL}, b_{iL}\}, \max\{a_{iU}, b_{iU}\}]$. In our earlier work [8], we have represented IVFM $A = (a_{ij}) = ([a_{ijL}, a_{ijU}])$ where each a_{ij} is a subinterval of the interval [0, 1], as the interval matrix

$$A = [A_L, A_U] \tag{1.1}$$

whose ijth entry is the interval $[a_{ijL}, a_{ijU}]$, where the lower limit $A_L = (a_{ijL})$ and the upper limit $A_U = (a_{ijU})$ are fuzzy matrices such that the $A_L \le A_U$ that is $a_{ijL} \le a_{ijU}$ under the usual ordering of real numbers. In this paper, we adopt a similar technique to determine the shortest path for an IVFN, that is, a path in which the sum of the weight of its constituent edges is minimized, by way of constructing two fuzzy networks corresponding to the lower and upper limits of an IVFN as a generalization of fuzzy shortest path technique presented in [4] and analogous to that for shortest path technique found in [1]. In section 2, we present the basic definition and notations. In section 3, we propose a new approach to determine the shortest path in IVFN in which the edges representing the roads connecting the cities and each edge (i, i+1) has an associated weight representing the traffic on the road connecting the cities i and i+1 , which is an interval fuzzy number of the form $R_i = [R_{iL}, R_{iU}]$ for each i and we apply the technique used in [1, 4] to determine the shortest path in lower and upper limits of the fuzzy networks. We have defined the shortest path for an IVFN as

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that path for which the shortest path in lower limit fuzzy network coincides with the shortest path in upper limit fuzzy network and weight = $[w_L, w_U]$ where the w_L and w_U are the weights of the shortest path for lower and upper limit fuzzy networks respectively.

2. PRELIMINARIES

In this section, basic definitions and notations are given.

Let a graph, denoted as (V, E), be a set of points V and a set of pairs of these points E. The set V refers to the vertices of the graph and the set E refers to the edges of the graph. An edge is denoted by a pair of vertices {i, j}.

If E is changed to a set of ordered pairs of distinct elements of V, then G (V, E) is a directed graph and E is the set of ordered pairs (i, j). The ordered pairs (i, j) are referred to as arcs or edges and an arc goes from vertex i to vertex j. An arc (i, i) is referred to as a loop. A path from a vertex s to a vertex t is a sequence of arcs of the form (s, i_1), (i_1, i_2)……. (i_k, t).

If each arc (i, j) has an associated weight or length C_{ij}, then an (s, t) path has an associated weight or length equal to sum of the weights of the constituent arcs in the path. This in turn gives rise to the shortest path problem, which is to find the path with minimal weight between two vertices s and t.

There are a variety of ways to find one shortest path for a network [6]. Some of the more general methods such as the labeling algorithm follow from dynamic programming. It is assumed that the graphs for the models to be presented or directed graphs, that is graph without cycles.

For an acyclic directed graph G (V, E) with N vertices numbered from 1 to N such that '1' is the source and 'N' is the sink, a dynamic programming (DP) formulation for the shortest path problem is given as in [1]:

$$f_i(S_{i+1}) = \min_{X_i} (R_i\,(X_i, S_{i+1}) + f_{i-1}(S_i)) \qquad (2.1)$$

Where $f_{i-1}(S_i)$ denotes the optimal value of the objective function corresponding to the last i-1 stages and S_i is the input to the stage i-1, X_i denotes the vector of decision variable at stage i, $R_i(X_i, S_{i+1})$ is the return function of the stage i and $f_i(S_{i+1})$ denotes the optimal value of the objective function corresponding to the last i stages and S_{i+1} is the input to the stage i. Through the algorithm, vertex i is labeled with f(i), and labels allow the determination of the path.

Through Belman's principle of optimality this recursion (2.1) is very flexible and has many applications. One obvious flexibility is that the sum in (2.1) can be replaced by almost any binary operator and the recursion will hold in [4]. For the fuzzy optimization problems under the max min composition, the sum in (2.1) is the fuzzy addition and (2.1) is reformulated as

$$f_i(S_{i+1}) = \min_{X_i} \{\max[R_i(X_i, S_{i+1}), f_{i-1}(S_i)]\} \qquad (2.2)$$

3. The shortest path of an IVFN.

An interval fuzzy network includes nodes and directed links. Each node represents a city. Each directed links (i, i+1) connects city i to i+1. Let $X_i = \{X_1, X_{2\ldots}\, X_{i-1}\}$ denotes the vector of decision variable at stage i and $S_i = \{S_1, S_{2\ldots}\, S_{i-1}\}$ is the input to stage i-1, f_{i-1} denotes the fuzzy optimal value of the objective function corresponding to the last i-1 stages.

If $X_i \xrightarrow{R_i} S_{i+1}$, then it indicates that the degree of relevance from stage i to stage i+1 is R_i where R_i is a sub interval of [0,1]. Let

$$R_i = [R_{iL}, R_{iU}] \qquad (3.1)$$

Since R_i is an interval of [0, 1], $R_{iL} \le R_{iU}$. $R_i\,(X_i, S_{i+1})$ is the weight of the corresponding arc (i, i+1). For this interval valued network (IVFN), let us construct two networks which we call as lower limit fuzzy network $(FN)_L$ and upper limit fuzzy network $(FN)_U$ with the same set of nodes and links, the weight of the corresponding arc (i, i+1) in the lower limit fuzzy network is R_{iL} and in the upper limit fuzzy network in R_{iU}.

This fuzzy shortest path networks can also be viewed in terms of the Dynamic programming (DP) recursion given in Equation (2.1). This recursion is very close to Ford's Algorithm and is easily extended to fuzzy numbers as in Equation (2.2). Then the DP recursion for lower fuzzy network is,

$$f_{iL}(S_{i+1}) = \min_{X_i} \{\max[R_{iL}(X_i, S_{i+1}), f_{(i-1)L}(S_i)]\} \tag{3.2}$$

Where $f_{(i-1)L}(S_i)$ denotes the optimal value of the objective function corresponding to the last i-1 stages and S_i is the input to the stage i-1 of lower fuzzy networks$(FN)_L$, X_i denotes the vector of decision variable at stage i, $R_{iL}(X_i, S_{i+1})$ is the return function of the stage i and $f_{iL}(S_{i+1})$ denotes the optimal value of the objective function corresponding to the last i stages and S_{i+1} is the input to the stage i of lower fuzzy networks$(FN)_L$.

DP recursion for upper fuzzy network is,

$$f_{iU}(S_{i+1}) = \min_{X_i} \{\max[R_{iU}(X_i, S_{i+1}), f_{(i-1)U}(S_i)]\} \tag{3.3}$$

Where $f_{(i-1)U}(S_i)$ denotes the optimal value of the objective function corresponding to the last i-1 stages and S_i is the input to the stage i-1 of upper fuzzy networks$(FN)_U$, X_i denotes the vector of decision variable at stage i, $R_{iU}(X_i, S_{i+1})$ is the return function of the stage i and $f_{iU}(S_{i+1})$ denotes the optimal value of the objective function corresponding to the last i stages and S_{i+1} is the input to the stage i of upper fuzzy networks$(FN)_L$.

Let us define DP recursion for Interval valued fuzzy network as,

$$f_{i-1}(S_i) = [f_{(i-1)L}(S_i), f_{(i-1)U}(S_i)] \tag{3.4}$$

Then by recursion

$$f_i(S_{i+1}) = [f_{iL}(S_{i+1}), f_{iU}(S_{i+1})] \tag{3.5}$$

By equation (3.2) and (3.3) we have,

$$\begin{aligned} f_i(S_{i+1}) &= [\min_{X_i} \{\max[R_{iL}(X_i, S_{i+1}), f_{(i-1)L}(S_i)]\}, \ \min_{X_i} \{\max[R_{iU}(X_i, S_{i+1}), f_{(i-1)U}(S_i)]\}] \\ &= [\min_{X_i} \{\max\{[R_{iL}(X_i, S_{i+1}), R_{iU}(X_i, S_{i+1})], [f_{(i-1)L}(S_i), f_{(i-1)U}(S_i)\}\}] \\ &= [\min_{X_i} \{\max[R_i(X_i, S_{i+1}), f_{i-1}(S_i)]\}] \quad \text{(By (1.1))} \end{aligned} \tag{3.7}$$

Where $f_i(S_{i+1})$ denotes the optimal value of the objective function corresponding to the last i stages and S_{i+1} is the input to the stage i of Interval valued fuzzy networks(IVFN), $f_{(i-1)}(S_i)$ denotes the optimal value of the objective function corresponding to the last i-1 stages and S_i is the input to the stage i-1 of Interval valued fuzzy networks(IVFN), X_i denotes the vector of decision variable at stage i, $R_i(X_i, S_{i+1})$ is the return function of the stage i of Interval valued fuzzy networks(IVFN) .

Definition 3.1:
Shortest path in IVFN = Shortest path in lower limit fuzzy network $(FN)_L$
= Shortest path in upper limit fuzzy network $(FN)_U$.

Weight of the shortest path of IVFN = $[w_L, w_U]$ where w_L and w_U are weights of the fuzzy shortest path in $(FN)_L$ and $(FN)_U$ respectively.

Now, we deal Interval valued fuzzy networks by using the algorithm [1], applied to $(FN)_L$ and $(FN)_U$ independently.

Algorithm 3.2:

Step 1: Identify the decision variables and specify objective function to be optimized for interval valued fuzzy networks.
Step 2: Decompose the network into a number of smaller sub intervals. Identify the stage variable at each stage and write down the fuzzy transformation function as a function of the state variable and decision variable at the next stage.
Step 3: Write down a general recursive relationship for completing the fuzzy optimal policy of IVFN by using the interval valued fuzzy dynamic programming recursion in (3.4) and (3.7).
Step 4: Construct appropriate stage to show the required values of the return function at each Stage in IVFN.
Step 5: Determine the overall fuzzy optimal decision or policy and its value at each stage of an IVFN.
Step 6: We get the shortest path of IVFN.

Now, A_N^T be the interval valued fuzzy networks, representing the weight of N during time interval T.

$$A_N^T = [A_{NL}^T, \; A_{NU}^T] \tag{3.8}$$

Where A_{NL}^T is the lower limit (R_{iL}) of the fuzzy network and A_{NU}^T is upper limit (R_{iU}) of the fuzzy network. Then,

$$\text{shortest path in } A_N^T = \text{ shortest path in } A_{NL}^T = \text{ shortest path in } A_{NU}^T \tag{3.9}$$

$$\text{Weight of the shortest path of IVFN} = [\text{weight of the shortest path in } A_{NL}^T, \text{ Weight of the shortest path in } A_{NU}^T] \tag{3.10}$$

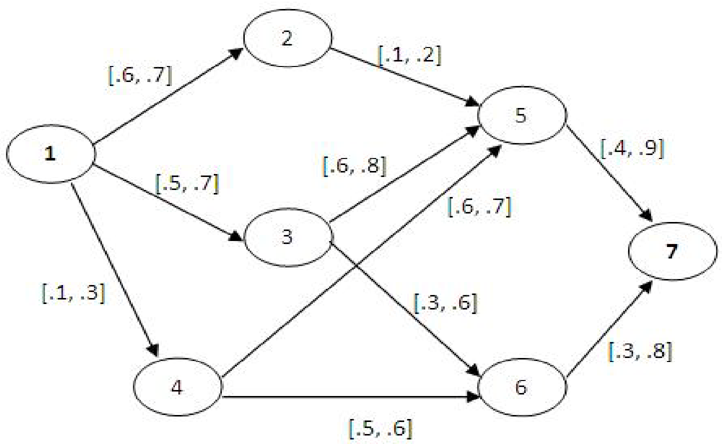
We shall illustrate the technique with a simple example and provide the mathematical verification.

Illustration 3.3:

We consider a Network N = (V, E) consisting n nodes (cities) and m edges (roads) connecting the cities of a country. If we measure the crowdness that is traffic of the roads of the network for particular time duration, it is quite impossible to measure the crowdness in duration as it is not fixed, but varies from time to time. So, appropriate technique to gradation of crowdness is an interval and not a point.

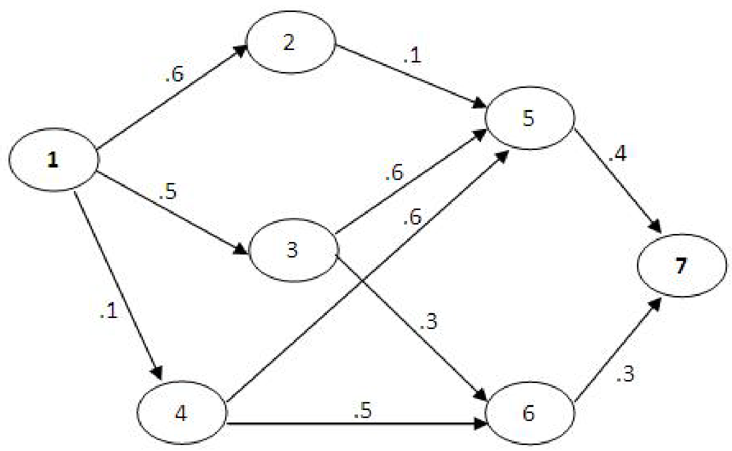
In this case, the network N is an interval valued fuzzy network in which the weight of the each arc (i, i+1) depends upon the crowdness.

Suppose that we want to select the shortest highway route (path) between two cities. The following route network provides the possible routes between the starting city at node 1 and the destination city at node 7. The routes pass through intermediate cities designated by nodes 2 to 6.



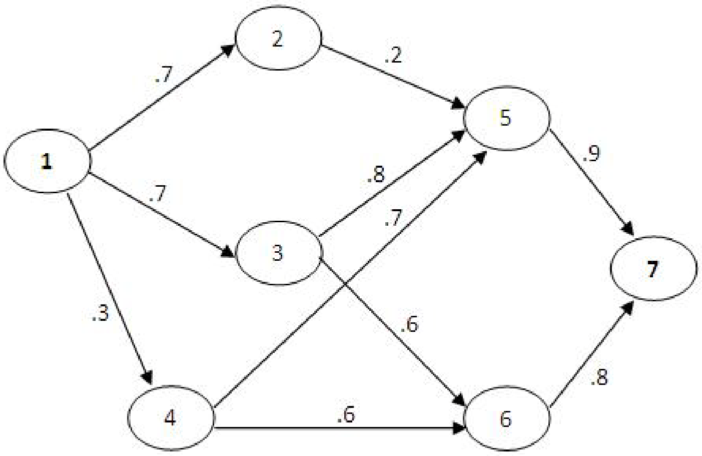
A_N^T- Internal valued fuzzy network (IVFN)

By using our representation (3.1) and (3.8), $A_N^T = [A_{NL}^T, \; A_{NU}^T]$



A_{NL}^T - lower limit fuzzy network $(FN)_L$

and

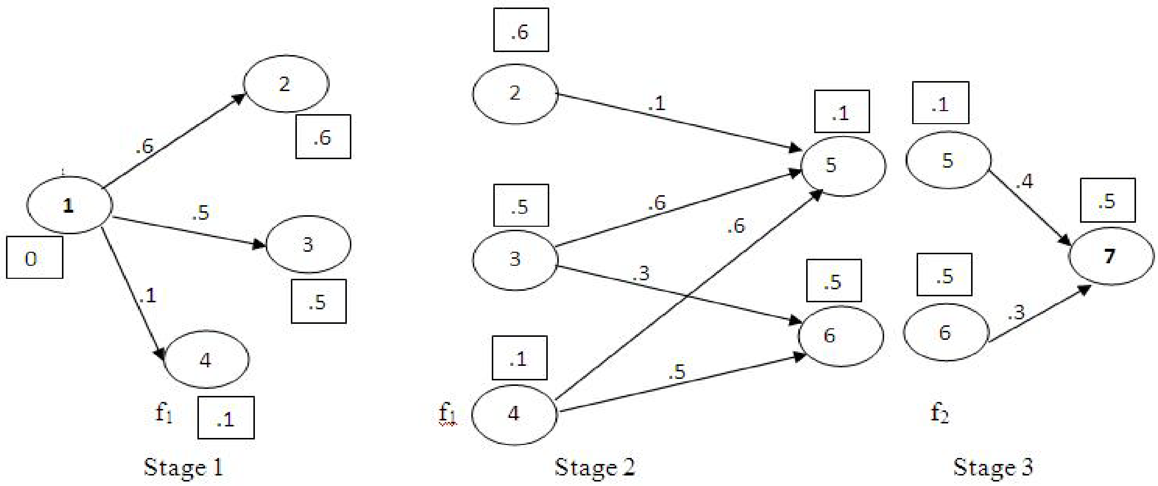


$A_{NU}{}^{T}$ - lower limit fuzzy network $(FN)_U$

Now we apply the algorithm (3.2) to find a path between city 1 to city 7 which is minimum among all the paths between city 1to city 7.

(i) Shortest path for the lower limit fuzzy network.

First decompose the lower limit fuzzy network into sub networks/stages as



Now S_1 is the state in which the node 1lies also, S_1 has only state value S_1=1. State S_2 has only three possible values; Say 2, 3, 4 corresponding stage 1, and so on. Possible alternative paths from one stage to the other will be called decision variables by X_i the decision which takes from S_{i-1} to S_i. The return or the gain which obviously being the function of decision will be denoted by $R_{iL}(X_i,S_{i+1})$. Here $R_{iL}(X_i, S_{i+1})$ can be identified with the lower limit of the corresponding arc. By equation (3.2) we have,

$$f_{iL}(S_{i+1}) = \min_{X_i}\{\max [R_{iL}(X_i, S_{i+1}),\ f_{(i-1)L}(S_i)]\}$$

Now initially for i = 0, $f_i(S_{i+1}) = f_0(S_1) = f_0(1) = 0$

For i =1 (stage1):

$$f_1(S_2) = \min_{X_i}\{\max [R_{iL}(X_1, S_2), f_0(S_1)]\}$$
$$= \min_{X_i}[R_{iL}(X_1, S_2)]$$

Now tabulating the date for f_1 (S_2)

S_1	S_2	X_1	R_{iL} (X_1, S_2)	$f_1(S_2)$	fuzzy optimal policy
1	2 3 4	1- 2 1- 3 1- 4	.6 .5 .1	.6 .5 .1	1- 2 1- 3 1- 4

For stage 2 (i = 2)

$$f_2\,(S_3) = \min_{X_2} \{\max\,[R_{2L}\,(X_2, S_3), f_1\,(S_2)]\}$$

S_2	S_3	X_2	R_{2L} (X_2,S_3)	max (R_2,f_1)	f_2 (S_3)	fuzzy optimal policy
2		2-5	.1	.6		2-5
3	5	3-5 3-6	.6 .3	.6 .5*	.5	3-5
4	6	4-5 4-6	.6 .5	.6 .5*	.5	4- 6

For last stage 3 (i = 3)

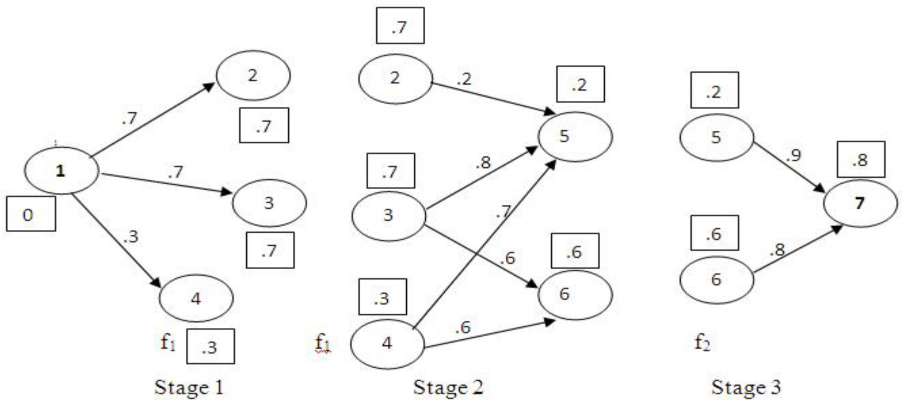
$$f_3(S_4) = \min_{X_3} \{\max\,[R_{3L}\,(X_3, S_4),\ f_2\,(S_3)]\}$$

S_3	S_4	X_3	R_{3L} (X_3,S_4)	max (R_3, f_2)	f_3	fuzzy optimal policy
5	7	5-7	.4	.6 .6 .6		
6		6-7	.3	.5* .5	.5	6-7

Therefore, for the lower limit fuzzy network of the shortest path from city 1to city7 is: 1 $\longrightarrow$ 4 $\longrightarrow$ 6 $\longrightarrow$ 7
Weight of the shortest path W_L = (.1, .5, .3)

(ii) Shortest path for the upper limit fuzzy matrices

Decompose the upper limit fuzzy network in to sub network /stages as follows:



Similarly we have to find the upper limit of the shortest path. Here R_{iU} (X_i, S_i+1) can be identified with the upper limit of the corresponding arc.

By equation (3.3) we have,

$$f_{iU}(S_i+1) = \min_{X_i} \{\max[R_{iU}(X_i, S_{i+1}), f_{(i-1)U}(S_i)]\}$$

Now, initially for i = 0, $f_i(S_{i+1}) = f_0(S_1) = f(1) = 0$

For i =1 (stage 1):

$$f_1(S_2) = \min_{X_1}\{\max [R_{1U}(X_1, S_2), f_0(S_1)]\}$$

$$= \min_{X_1} [R_{1U}(X_1, S_2)]$$

Now tabulating the data for $f_1(S_2)$

S_1	S_2	X_1	$R_{1U}(X_1, S_2)$	$f_1(S_2)$	fuzzy optimal policy
	2	1- 2	.7	.7	1- 2
1	3	1- 3	.7	.7	1- 3
	4	1- 4	.3*	.3*	1- 4

For stage 2 (i = 2):

$$f_2(S_3) = \min_{X_2} \{\max [R_2(X_2, S_3), f_1(S_2)]\}$$

S_2	S_3	$R_{2U}(X_2, S_3)$	max (R_2, f_1)	$f_2(S_3)$	fuzzy optimal policy
2		2-5	.7		2-5
3	5	3-5 3-6	.8 .7*	.7	3-6
4	6	4-5 4-6	.7 .6*	.6	4-6

For last stage 3 (i = 3)

$$f_3(S_4) = \min_{X_3} \{\max [R_{3u}(X_3, S_4), f_2(S_3)]\}$$

S_3	S_4	X_3	$R_{3L}(X_3,S_4)$	max (R_3,f_2)	F_3	fuzzy optimal policy
5		5-7	.9	.9 .9 .9		
6	7	6-7	.8	.8* .8	.8	6-7

Therefore the shortest path from city 1 to city 7 for the upper limit fuzzy network is: $1 \longrightarrow 4 \longrightarrow 6 \longrightarrow 7$

Weight of the shortest path w_U = (.3, .6, .8)

Now we conclude by equation (3.9) we have,

Shortest path in A_N^T = shortest path in A_{NL}^T = shortest path in A_{NU}^T = $1 \rightarrow 4 \rightarrow 6 \rightarrow 7$

By equation (3.10) we have,

Weight of the shortest path of IVFN = [weight of the shortest path in A_{NL}^T, weight of the shortest path in A_{NU}^T]

That is, $w = [w_L, w_U]$

= [(.1, .5, .3), (.3, .6, .8)]

= [.1, .3], [.5, .6], [.3, .8], [.1, .3] [.5, .6] [.3, .8]

Therefore shortest path of IVFN is $1 \rightarrow 4 \rightarrow 6 \rightarrow 7$.

CONCLUSION

For a given IVFN, we have constructed two fuzzy networks $(FN)_L$ and $(FN)_U$ with the associated weight R_{iL} and R_{iU} respectively. Since the vertex sets and edge sets are same for IVFN, $(FN)_L$ and $(FN)_U$ and weight of the each edge (i, i+1) in IVFN is an interval of the form $w_i = [w_{iL}, w_{iU}]$, we conclude that the shortest path for an IVFN is that path for which the shortest path in lower limit fuzzy network coincides with the shortest path in upper limit fuzzy network and weight = $[w_L, w_U]$ where the w_L and w_U are the weights of the shortest path for lower and upper limit fuzzy networks.

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