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On Gαo-kernel in the digital plane

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ABSTRACT

*We introduce the concept of *gα-closed sets in a topological space and characterize it using its Gαo-kernel. Moreover we investigate new seperation axioms and new functions in topological spaces. For the digital plane, we have explicite forms of Gαo-kernel and α-kernel of a subset in the plane.*

Key words: **gα-closed sets, $_\alpha T_{1/2}{}^{**}$ spaces, $T_c{}^{**}$ spaces, $_\alpha T_c{}^{**}$ spaces, $^{**}{}_\alpha T_{1/2}$ spaces, *gα-continuous, *gα-irresolute maps, *gαc- homeomorphism, Gαo-kernel, α-kernel and digtal plane.*

AMS Subject Classification: *54A05, 54C05, 54C10, 68U05, 68U10.*

1. INTRODUCTION

Levine [14] and Njastad [19] introduced semi-open sets and α-sets respectively. The complement of a semi-open (resp. α-open) set is called a semi-closed [3] (resp.α-closed [19]) set. Levine [13] introduced g-closed sets and studied their most fundamental properties. S.P. Arya and T. Nour [1], H. Maki et.al. [16, 17] introduced gs-closed sets, αg-closed sets and gα-closed sets respectively. Dontchev [9] and Gnanambal [10] introduced gsp-closed sets and gpr-closed sets respectively.

In this paper, we introduce a new class of sets, namely *gα-closed sets by generalizing gα-open sets. This new class is properly placed between the class of closed sets and the class of g-closed sets. Applying *gα-closed sets, we introduce and study some new spaces, namely $_\alpha T_{1/2}{}^{**}$ spaces, $T_c{}^{**}$ spaces, $_\alpha T_c{}^{**}$ spaces and $^{**}{}_\alpha T_{1/2}$ spaces. In the fifth chapter we introduce and study *gα-continuous, *gα-irresolute maps and its group structure., In the sixth chapter we investigate *gαc- homeomorphism and its properties. In the seventh chapter, we investigate the explicite form in the digital plane of Gαo-kernel and α-kernel which are used for charaterization of *gα-closed sets and gα-closed sets, respectively. The digital plane is a mathematical model of the computer screen (cf.[5],[11],[12]).

2. PRELIMINARIES

Throughout this paper (X, τ), (Y,σ) and (Z,η) represent topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), cl(A), int(A) and C(A) denote the closure of A, the interior of A and the complement of A in X respectively.

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1: A subset A of a space (X, τ) is called
1. a *semi-open* set [14] if A ⊆ cl(int(A)) and a *semi-closed* set if int(cl(A)) ⊆ A,
2. an *α-open* set [19] if A ⊆ int(cl(int((A))) and an *α-closed* set if cl(int(cl(A))) ⊆ A and

Definition 2.2: A subset A of a space (X, τ) is called
1. a *generalized closed* (briefly *g-closed*) set [13] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ). The complement of a g- closed set is called a g-open set,
2. a *generalized semi-closed* (briefly *gs-closed*) set [1] if scl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ),
3. an *α-generalized closed* (briefly *αg-closed*) set [16] if αcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ). The complement of an αg-closed set is called an αg-open set,
4. a *generalized α-closed* (briefly *gα-closed*) set [17] if αcl(A) ⊆ U whenever A ⊆ U and U is α-open in (X, τ),
5. a *generalized preclosed* (briefly *gp-closed*) set [18] if pcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ),
6. a *generalized semi-preclosed* (briefly *gsp-closed*) set [9] if spcl(A) ⊆ U whenever A ⊆ U and U is open in (X , τ),
7. a *generalized preregular closed* (briefly *gpr-closed*) set [10] if pcl(A) ⊆ U whenever A ⊆ U and is regular open in (X, τ),

Definition 2.3: A function f: (X, τ) →(Y,σ) is called
1. semi-continuous [14] if $f^{-1}(V)$ is semi-open in (X, τ) for every closed set V of (Y,σ),
2. α-continuous [15] if $f^{-1}(V)$ is α-closed in (X, τ) for every closed set V of (Y,σ),
3. g-continuous [2] if $f^{-1}(V)$ is g-closed in (X, τ) for every closed set V of (Y,σ),
4. gs-continuous [7] if $f^{-1}(V)$ is gs-closed in (X, τ) for every closed set V of (Y,σ),
5. αg-continuous [4] if $f^{-1}(V)$ is αg-closed in (X, τ) for every closed set V of (Y,σ),
6. gα-continuous [17] if $f^{-1}(V)$ is gα-closed in (X, τ) for every closed set V of (Y,σ),
7. gsp-continuous [9] if $f^{-1}(V)$ is gsp-closed in (X, τ) for every closed set V of (Y,σ),
8. gpr-continuous [10] if $f^{-1}(V)$ is gpr-closed in (X, τ) for every closed set V of (Y,σ),
9. gc-irresolute [2] if $f^{-1}(V)$ is g-closed in (X, τ) for every g-closed set V of (Y,σ),
10. gs-irresolute [7] if $f^{-1}(V)$ is gs-closed in (X, τ) for every gs-closed set V of (Y,σ),
11. αg -irresolute [4] if $f^{-1}(V)$ is αg -closed in (X, τ) for every αg -closed set V of (Y,σ) and
12. gα-irresolute [17] if $f^{-1}(V)$ is gα -closed in (X, τ) for every gα -closed set V of (Y,σ).

Definition 2.4:A space (X, τ) is called
1. a $T_{1/2}$ *space* [13] if every g-closed set is closed,
2. a T_b *space* [6] if every gs-closed set is closed,
3. a T_d *space* [6] if every gs-closed set is g-closed,
4. an $_{\alpha}T_b$ *space* [4] if every αg-closed set is closed,
5. an $_{\alpha}T_d$ *space* [4] if every αg-closed set is g-closed.

Notation 2.5: For a space (X, τ), C(X, τ) (resp.SC(X, τ), αC(X, τ), GαC(X, τ),GC(X, τ),GSC(X, τ), αGC(X, τ)) denote the class of all closed (resp.semi-closed, α-closed, gα-closed, g-closed, gs-closed, αg-closed) subsets of (X, τ).

3. BASIC PROPERTIES OF *gα-CLOSED SETS

We introduce the following definition.

Definition 3.1: A subset A of (X, τ) is called a *gα-closed set if cl(A) ⊆ U whenever A ⊆ U and U is gα-open in (X, τ).

The class of *gα-closed subsets of (X, τ) is denoted by *GαC(X, τ).

Theorem 3.2: Every closed set is a *gα -closed set.

Proof: Let A⊆U, where U is gα-open set in X. Since A is closed, cl(A) = A⊆U. Therefore cl(A) ⊆U.

Hence A is *gα-closed.

Following example shows that the above implication is not reversible.

Example 3.3: Let X = {a, b, c} and τ = {X, ϕ, {a, b}}. *GαC(X, τ) = {X, ϕ, {c}, {b, c}, {a, c}}.

Here{b, c}is a *gα-closed set of (X, τ) but it is not a closed set of (X, τ).

Theorem 3.4: Every *gα -closed set is g-closed set.

Proof: Let A⊆U, where U is an open set in X. Since every open set is gα-open, U is gα-open .Since A is *gα-closed, cl(A) ⊆U. Hence A is g-closed.

Following example shows that the above implication is not reversible.

Example 3.5: Let X = {a, b, c} and τ = {X, ϕ, {a}, {b, c}}. GC(X, τ) = {X, ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}}. *GαC(X, τ) = {X, ϕ, {a}, {b, c}}.

Here {b} is a g-closed set of (X, τ) it is not a *gα-closed set of (X, τ).

Theorem 3.6: Every *gα-closed set is gα-closed set.

Proof: Let A⊆U, where U is an α-open set in X. Since every α-open set is gα-open, U is gα-open. Since A is *gα-closed, cl(A) ⊆U. But αcl(A) ⊆ cl(A) ⊆U. Therefore A is gα-closed.

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Following example shows that the above implication is not reversible.

Example 3.7: Let X and τ be as in the example 3.5. Let A = {a, c}. A is a gα-closed set of (X, τ). But A is not a *gα-closed set of (X, τ).

Theorem 3.8: Every *gα-closed set is gp-closed set.

Proof: Let A⊆U, where U is an open set in X. Since every open set is gα-open, U is gα-open. Since A is *gα-closed, cl(A) ⊆U. But pcl(A) ⊆ cl(A) ⊆U. Therefore A is gp-closed set.

Following example shows that the above implication is not reversible.

Example 3.9: Let X and τ be as in the example 3.5. Let B= {a, b}. B is a gp-closed set of (X, τ). But B is not a *gα-closed set of (X, τ).

Thus the class of *gα-closed sets are contained in the class of g-closed sets, gα-closed sets, αg-closed sets, gs-closed sets, gsp-closed sets, gpr-closed sets and gp-closed sets. The class of *gα-closed sets contains the class of closed sets.

Remark 3.10: *gα-closedness is independent of semi-closedness and α-closedness.

Proof: It can be seen by the following example.

Example 3.11: Let X = {a, b, c} and τ = {X, ϕ, {a}}. SC(X, τ) = {X, ϕ, {b}, {c}, {b, c}} = αC(X, τ)
*GαC(X, τ) = {X, ϕ, {b, c}}.

Here {b} is semi-closed set and α-closed set of (X, τ). But it is not a *gα-closed set of (X, τ).

Example 3.12: Let X and τ be as in the example 3.3. Here {b, c} is not a semi-closed and α-closed set of (X, τ). But it is a *gα-closed set of (X, τ).

Theorem 3.13: The intersection of two gα-closed sets is again in gα-closed set.

Proof: Let A and B are gα-closed sets. Let A∩B ⊆ U, U is α-open. Since A and B are gα-closed sets, αcl(A) ⊆ U and αcl(B) ⊆ U. This implies that αcl(A∩B) = αcl(A) ∩ αcl(B) ⊆ U ⇒ αcl(A∩B) ⊆ U. Therefore A∩B is gα-closed.

Theorem 3.14: Let A be an open set and B be an gα-open set, then A∪B is gα-open set.

Proof: Suppose that A is an open set and B is an gα-open set. Since every open set is gα-open set, A is gα-open set. Then A∪B is gα-open set, since union of two gα-open set is again gα-open set.

Theorem 3.15:
1. Let A be a *gα-closed set of (X, τ) if and only if cl(A)-A does not contain any non empty gα-closed set.

2. If A is a *gα-closed and A ⊆ B ⊆ cl(A), then B is *gα-closed.

Proof:
1. Necessity part- Suppose that A is *gα-closed and let F be a non empty gα-closed set with F ⊆ cl(A)-A. Then A ⊆ X-F and so cl(A) ⊆ X-F. Hence F ⊆ X-cl(A), a contradiction.

Sufficient part - Suppose A is a subset of (X, τ) such that cl(A)-A does not contain any non-empty gα-closed set. Let U be a gα-open set of (X,τ) such that A ⊆ U. If cl(A)⊆ U, then cl(A)∩C(U) ≠ ϕ.Then ϕ ≠ cl(A)∩C(U) is a gα-closed set of (X, τ), since the intersection of two gα-closed sets is again gα-closed set.

2. Let U be a gα-open set of (X, τ) such that B ⊆ U. Then A ⊆ U. Since A is *gα-closed, cl(A) ⊆ U. Now cl(B) ⊆ cl(cl(A))=cl(A) ⊆ U. Therefore B is also a *gα-closed set of (X, τ).

Theorem 3.16: Let X be a topological space. A subset A of (X, τ) is *gα-open if and only if U⊆Int(A), whenever U is gα-closed set and U⊆ A.

Proof: Let A be a *gα-open set and U is gα-closed set such that U⊆ A implies X-U ⊇ X-A and X-A is *gα-closed set. So cl(X-A) ⊆ X-U implies (X- cl(X-A)) ⊇ (X-(X-U)) = U. But (X- cl(X-A)) = Int(A). Thus U⊆Int(A).

Conversely, suppose A is subset such that U⊆Int(A). Whenever U is gα-closed and U⊆ A. We show that X-A is *gα-closed set. Let X-A ⊆ U, where U is gα-open. Since X-A ⊆ U implies X-U ⊆ A. By assumption that we must have X-U ⊆Int(A) or X-Int(A) ⊆ U. Now X-Int(A) = cl(X-A) which implies that cl(X-A) ⊆ U and X-A is *gα-closed set.

Theorem 3.17: The union of two *gα-closed sets is a *gα-closed set.

Proof: Let A and B are *gα-closed sets. Let A∪B ⊆U, U is gα-open. Since A and B are *gα-closed sets, cl(A)⊆U and cl(B)⊆U. This implies that cl(A∪B) = cl(A) ∪ cl(B) ⊆U ⇒ cl(A∪B) ⊆U. Therefore A∪B is *gα-closed.

Remark 3.18: The intersection of two *gα-closed sets is again in *gα-closed set.
(i) The intersection of two *gα-closed sets is again in *gα-closed set.
(ii) The intersection of an open and a *gα-open sets is a *gα-open set.
(iii) The union of an open and a *gα-open sets is a *gα-open set.

We prepare the following notations:

For a subset A of (X, τ),
GαO(X, τ) = {U/U is gα-open in (X, τ)};
ker(A) = ∩{U/U∈τ and A ⊆ U};
α-ker(A) = ∩{U/U is α-open set and A ⊆ U};
GαO-ker(A) = ∩{U/U∈ GαO(X, τ) and A ⊆ U}.
$X_{g\alpha c}$ = {x∈X / {x} is gα-closed in (X, τ)} and
$X_{*g\alpha o}$ = {x∈X / {x} is *gα-open in (X, τ)}.

Theorem 3.19: Any subset A is gα-closed set if and only if αcl(A) ⊆ α-ker(A) holds.

Proof: Necessary: We know that A ⊆ α-ker(A). Since A is gα-closed, then αcl(A) ⊆ α-ker(A).

Sufficiency: Let A ⊆ U and U is α-open. Given that αcl(A) ⊆ α-ker(A). If U ⊆ αcl(A), then α-ker(A) ⊆ U ⊆ αcl(A), which is a contradiction to the hypothesis. Therefore αcl(A) ⊆ U. Hence A is gα-closed.

Lemma 3.20: For any space (X, τ), $X = X_{g\alpha c} \cup X_{*g\alpha o}$ holds.

Proof: Let x∈X. Suppose that {x} is not *gα-closed set in (X, τ). Then X is a unique gα-open set containing X/{x}. Thus X/{x} is *gα-closed in (X, τ) and so {x} is *gα-open. Therefore $x \in X_{g\alpha c} \cup X_{*g\alpha o}$.

Theorem 3.21: For a subset A of (X, τ), the following conditions are equivalent:
1. A is *gα-closed in (X, τ).
2. cl(A) ⊆ GαO-ker(A) holds.
3. (i) $cl(A) \cap X_{g\alpha c} \subseteq A$ and (ii) $cl(A) \cap X_{*g\alpha o} \subseteq$ GαO-ker(A) holds.

Proof:
(1)⇒(2): Let x∉ GαO-ker(A). Then there exists a set U∈ GαO(X, τ) such that x∉U and A ⊆ U.

Since A is *gα-closed, cl(A) ⊆ U and x∉cl(A). This is a contradiction.

(2)⇒(3):

(i): It follows from (2) that $cl(A) \cap X_{g\alpha c} \subseteq$ GαO-ker(A) $\cap X_{g\alpha c}$. We claim that GαO-ker(A) $\cap X_{g\alpha c} \subseteq$ A.Suppose x ∈ GαO-ker(A) $\cap X_{g\alpha c}$ and assume that x∉A.Since the set X/{x}∈ GαO(X, τ) and A ⊆ X/{x}. Then we have that x ∈ X/{x} and so this is a contradiction. Thus we show that $cl(A) \cap X_{g\alpha c} \subseteq$ A. by using (2) $cl(A) \cap X_{g\alpha c} \subseteq$ GαO-ker(A) $\cap X_{g\alpha c} \subseteq$ A.

(ii): It is obtained by (2).

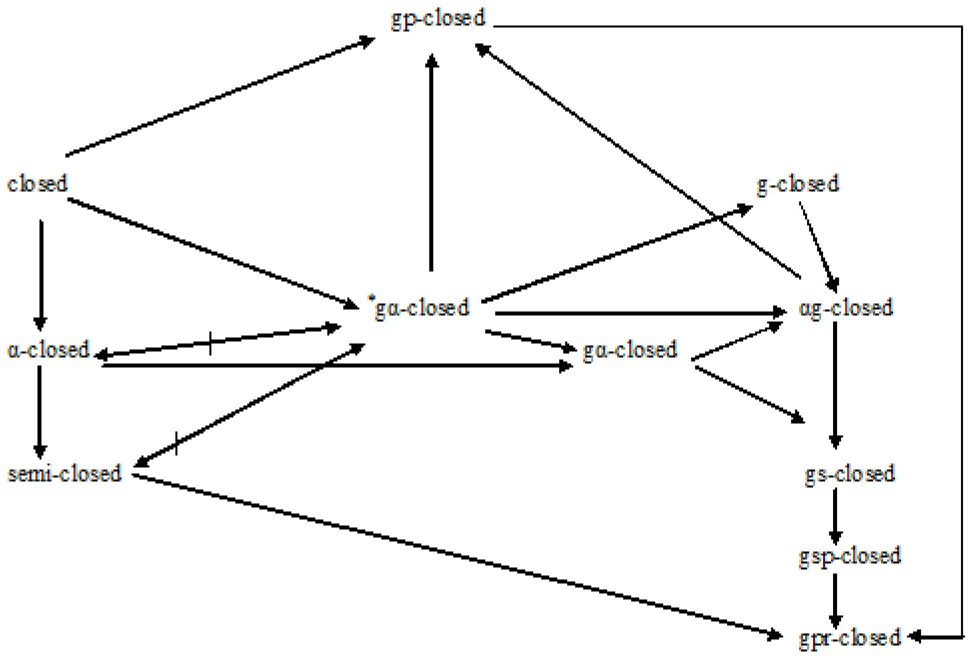
(3) ⇒(2): By Remark 3.8 and (3),

$$\begin{aligned} cl(A) = cl(A) \cap X &= cl(A) \cap (X_{g\alpha c} \cup X_{*g\alpha o}) \\ &= (cl(A) \cap X_{g\alpha c}) \cup (cl(A) \cap X_{*g\alpha o}) \\ &\subseteq A \cup \text{G}\alpha\text{O-ker}(A) \\ &= \text{G}\alpha\text{O-ker}(A). \end{aligned}$$

That is cl(A) ⊆ GαO-ker(A) holds.

(2) ⇒ (1): Let U∈ GαO(X, τ) such that A ⊆ U. Then we have that GαO-ker(A) ⊆ U and so by (2) cl(A) ⊆ U. Therefore A is *gα-closed.

Remark 3.22: The following diagram shows the relationships established between *gα-closed sets and some other sets in theorem 3.2, 3.4, 3.6, 3.8, remark3.10 and reference [22], [21]. A→B (A ↮ B) represents A implies B but not conversely(A and B are independent each other).



4. APPLICATIONS OF *gα-CLOSED SETS

We introduce the following definition.

Definition 4.1: A space (X, τ) is called an ${}_{\alpha}T_{1/2}^{**}$ space if every *gα-closed set is closed.

The following theorem gives a characterization of ${}_{\alpha}T_{1/2}^{**}$ spaces.

Theorem 4.2: If (X, τ) is an ${}_{\alpha}T_{1/2}^{**}$ space, then every singleton of X is either gα-closed or open.

Proof: Let x ∈ X and suppose that {x} is not a gα-closed set of (X, τ). Then X/{x} is not gα-open. This implies that X is the only gα-open set containing X/{x}, so X/{x} is a *gα-closed set of (X, τ).Since (X, τ) is an ${}_{\alpha}T_{1/2}^{**}$ space, X/{x} is closed or equivalently {x} is open in (X, τ).

Theorem 4.3: Every $T_{1/2}$ space is an ${}_{\alpha}T_{1/2}^{**}$ space.

Proof: Let A be a *gα-closed set of (X, τ). Since every *gα-closed set is g-closed, A is g-closed. Since (X, τ) is a $T_{1/2}$ space, A is closed. Therefore (X, τ) is an ${}_{\alpha}T_{1/2}^{**}$ space.

The space in the following example is an ${}_{\alpha}T_{1/2}^{**}$ space but not a $T_{1/2}$ space.

Example 4.4: Let X={a, b, c}with τ ={X, ϕ,{a},{a, b}}.
*GαC(X, τ)={X, ϕ,{c}, {b, c}}
GC(X, τ) = {X, ϕ,{c}, {b, c}, {a, c}}.

Here (X, τ) is an ${}_{\alpha}T_{1/2}^{**}$ space but not a $T_{1/2}$ space. Since {a, c} is a g-closed set but not a closed set.

Theorem 4.5: Every T_b space is an ${}_{\alpha}T_{1/2}^{**}$ space.

Proof: Let A be a $^*g\alpha$-closed set of (X, τ). Since every $^*g\alpha$-closed set is gs-closed, A is gs-closed. Since (X, τ) is a T_b space, A is closed. Therefore (X, τ) is an ${}_\alpha T_{1/2}^{**}$ space.

The space in the following example is an ${}_\alpha T_{1/2}^{**}$ space but not a T_b space.

Example 4.6: Let X= {a, b, c} with τ ={X, ϕ,{b}, {b, c}}. $^*G\alpha C$(X, τ) ={X, ϕ,{a}, {a, c}}
GSC(X, τ) = {X, ϕ,{a}, {c}, {a, b}, {a, c}}.

Here (X, τ) is an ${}_\alpha T_{1/2}^{**}$ space but not a T_b space. Since {a, b} is a gs-closed set but not a closed set.

Theorem 4.7: Every ${}_\alpha T_b$ space is an ${}_\alpha T_{1/2}^{**}$ space.

Proof: Let A be a $^*g\alpha$-closed set of (X, τ). Since every $^*g\alpha$-closed set is αg-closed, A is αg-closed. Since (X, τ) is an ${}_\alpha T_b$ space, A is closed. Therefore (X, τ) is an ${}_\alpha T_{1/2}^{**}$ space.

The space in the following example is an ${}_\alpha T_{1/2}^{**}$ space but not an ${}_\alpha T_b$ space.

Example 4.8: Let X and τ be as in example 4.6. Here (X, τ) is an ${}_\alpha T_{1/2}^{**}$ space but not an ${}_\alpha T_b$ space. Since {c}is an αg-closed set but not a closed set.

Definition 4.9: A space (X, τ) is called a T_c^{**} if every gs-closed set is $^*g\alpha$-closed.

The following theorem gives a characterization of T_c^{**} spaces.

Theorem 4.10: If (X, τ) is a T_c^{**} space, then every singleton of X is either closed or $^*g\alpha$-open.

Proof: Let $x \in X$ and suppose that {x} is not a closed set of (X, τ). Then X/{x} is not open. This implies X is the only open set containing X/{x}. So X/{x} is a gs-closed set of (X, τ). Since (X, τ) is a T_c^{**} space, X/{x} is a $^*g\alpha$-closed set or equivalently {x} is $^*g\alpha$-open in (X, τ).

The converse of the above theorem is not true as can be seen by the following example.

Example 4.11: Let X = {a, b, c} with τ = {X, ϕ,{a},{b},{a, b}}.
$^*g\alpha$-open sets of (X, τ) are X, ϕ,{a},{b},{a, b}.
GSC(X, τ) ={X, ϕ,{a},{b},{c},{b, c},{a, c}}.
$^*G\alpha C$(X, τ) = {X, ϕ,{c},{b, c},{a, c}}.

Here {a} and {b} are $^*g\alpha$-open sets and {c} is a closed set but (X, τ) is not a T_c^{**} space. Since {b} is a gs-closed set but not a $^*g\alpha$-closed set of (X, τ).

Theorem 4.12: Every T_b space is a T_c^{**} space.

Proof: Let A be a gs-closed set of (X, τ). Since (X, τ) is a T_b space, A is closed. Since every closed set is $^*g\alpha$-closed, A is $^*g\alpha$-closed set. Therefore (X, τ) is a T_c^{**} space.

The space in the following example is a T_c^{**} space but not a T_b space.

Example 4.13: Let X and τ be as in example 3.3. Here (X, τ) is a T_c^{**} space but not a T_b space. Since {a, c} is a gs-closed set but not a closed set.

Theorem 4.14: Every T_c^{**} space is a T_d space.

Proof: Let A be a gs-closed set of (X, τ). Since (X, τ) is a T_c^{**} space, A is $^*g\alpha$- closed. Since every $^*g\alpha$ -closed set is g-closed, A is g-closed set. Therefore (X, τ) is a T_d space.

The space in the following example is a T_d space but not a T_c^{**} space.

Example 4.15: Let X and τ be as in example 3.5. Here (X, τ) is a T_d space but not a T_c^{**} space. Since {b} is a gs-closed set but not $^*g\alpha$-closed set.

Theorem 4.16: Every T_c^{**} space is an ${}_\alpha T_d$ space.

Proof: Let A be a αg -closed set of (X, τ). Since every αg -closed set is gs-closed, A is gs-closed. Since (X, τ) is a T_c^{**} space, A is *gα- closed. Since every *gα -closed set is g-closed, A is g-closed set. Therefore (X, τ) is an $_\alpha T_d$ space.

The space in the following example is an $_\alpha T_d$ space but not a T_c^{**} space.

Example 4.17: Let X and τ be as in example 3.5. Here (X, τ) is an $_\alpha T_d$ space but not a T_c^{**} space. Since {a, c} is a gs-closed set but not *gα-closed set.

Theorem 4.18: The space (X, τ) is a T_b space if and only if it is a T_c^{**} space and an $_\alpha T_{1/2}^{**}$ space.

Proof: Necessity part: By theorem 4.12 and 4.5.

Sufficient part: Let A be a gs-closed sets of (X, τ). Since (X, τ) is a T_c^{**} space, A is *gα-closed set. Since (X, τ) is an $_\alpha T_{1/2}^{**}$ space, A is closed. Therefore (X, τ) is an T_b space.

Remark 4.19: T_c^{**}space and $_\alpha T_{1/2}^{**}$ space are independent of each other.

It can be seen by the following examples.

Example 4.20: Let X and τ be as in example 3.5. Here (X, τ) is an $_\alpha T_{1/2}^{**}$ space but not a T_c^{**}space. Since {b}is gs-closed set but not *gα-closed set.

Example 4.21: Let X and τ be as in example 3.3. Here (X, τ) is a T_c^{**} space but not an $_\alpha T_{1/2}^{**}$ space. Since {b, c}is *gα -closed set but not closed set.

Definition 4.22: A space (X, τ) is called an $_\alpha T_c^{**}$space if every αg-closed set is *gα-closed.

Theorem 4.23: Every T_b space is an $_\alpha T_c^{**}$space.

Proof: Let A be a αg-closed set of (X, τ). Since every αg-closed set is gs-closed, A is gs-closed. Since (X, τ) is a T_b space, A is closed. Since every closed set is *gα-closed, A is *gα-closed set. Therefore (X, τ) is a $_\alpha T_c^{**}$space.

The space in the following example is an $_\alpha T_c^{**}$space but not a T_b space.

Example 4.24: Let X and τ be as in example 3.3. Here (X, τ) is an $_\alpha T_c^{**}$ space but not a T_b space. Since {b, c} is a gs-closed set but not closed set.

Theorem 4.25: Every $_\alpha T_b$ space is an $_\alpha T_c^{**}$space.

Proof: Let A be a αg-closed set of (X, τ). Since (X, τ) is a $_\alpha T_b$ space, A is closed. Since every closed set is *gα-closed, A is *gα-closed set. Therefore (X, τ) is an $_\alpha T_c^{**}$space.

The space in the following example is an $_\alpha T_c^{**}$space but not an $_\alpha T_b$ space.

Example 4.26: Let X and τ be as in example 3.3. Here (X, τ) is an $_\alpha T_c^{**}$space but not an $_\alpha T_b$ space. Since {a, c} is a αg-closed set but not closed set.

Theorem 4.27: Every $_\alpha T_c^{**}$space is an $_\alpha T_d$ space.

Proof: Let A be a αg-closed set of (X, τ). Since (X, τ) is a $_\alpha T_c^{**}$ space, A is *gα -closed. Since every *gα -closed set is g-closed, A is g-closed set. Therefore (X, τ) is an $_\alpha T_d$ space.

The space in the following example is an $_\alpha T_d$ space but not an $_\alpha T_c^{**}$space.

Example 4.28: Let X and τ be as in example 3.5. Here (X, τ) is an $_\alpha T_d$ space but not an $_\alpha T_c^{**}$space. Since {c} is a αg-closed set but not *gα-closed set.

Theorem 4.29: Every T_c^{**}space is an $_\alpha T_c^{**}$ space.

Proof: Let A be a αg-closed set of (X, τ). Since every αg-closed set is gs-closed, A is gs-closed. Since (X, τ) is a T_c^{**} space, A is *gα-closed. Therefore (X, τ) is an $_\alpha T_c^{**}$space.

The space in the following example is an $_{\alpha}T_c^{**}$ space but not a T_c^{**} space.

Example 4.30: Let X and τ be as in example 4.11. Here (X, τ) is an $_{\alpha}T_c^{**}$ space but not a T_c^{**} space. Since{a} is a gs-closed set but not *gα-closed set.

Theorem 4.31: The space (X, τ) is an $_{\alpha}T_b$ space if and only if it is a $_{\alpha}T_c^{**}$ space and an $_{\alpha}T_{1/2}^{**}$ space.

Proof: Necessity part: By theorem 4.25 and 4.7.

Sufficient part: Let A be a αg-closed set of (X, τ). Since (X, τ) is an $_{\alpha}T_c^{**}$ space, A is *gα-closed. Since (X, τ) is an $_{\alpha}T_{1/2}^{**}$, A is closed set. Therefore (X, τ) is an $_{\alpha}T_b$ space.

Remark 4.32: $_{\alpha}T_c^{**}$ space and $_{\alpha}T_{1/2}^{**}$ space are independent of each other.

It can be seen by the following examples.

Example 4.33: Let X and τ be as in example 3.5. Here (X, τ) is an $_{\alpha}T_{1/2}^{**}$ space but not an $_{\alpha}T_c^{**}$ space. Since {b}is αg-closed set but not *gα-closed set.

Example 4.34: Let X and τ be as in example 3.3.Here (X, τ) is an $_{\alpha}T_c^{**}$ space. But not an $_{\alpha}T_{1/2}^{**}$ space. Since {b, c}is *gα -closed set but not closed set.

Definition 4.35: A space (X, τ) is called a $^{**}{}_{\alpha}T_{1/2}$ space if every g-closed set is *gα-closed set.

Theorem 4.36: Every $T_{1/2}$ space is a $^{**}{}_{\alpha}T_{1/2}$ space.

Proof: Let A be a g-closed set of (X, τ). Since (X, τ) is a $T_{1/2}$ space, A is closed. Since every closed set is *gα-closed, A is *gα-closed. Therefore (X, τ) is an $^{**}{}_{\alpha}T_{1/2}$ space.

The space in the following example is a $^{**}{}_{\alpha}T_{1/2}$ space but not a $T_{1/2}$ space.

Example 4.37: Let X and τ be as in example 3.3. Here (X, τ) is a $^{**}{}_{\alpha}T_{1/2}$ space but not a $T_{1/2}$ space. Since {b, c} is a g-closed set but not closed set.

Theorem 4.38: Every T_b space is a $^{**}{}_{\alpha}T_{1/2}$ space.

Proof: Let A be a g-closed set of (X, τ). Since every g-closed set is gs-closed, A is gs-closed set. Since (X, τ) is an T_b space, A is closed. Since every closed set is *gα-closed, A is *gα-closed. Therefore (X, τ) is an $^{**}{}_{\alpha}T_{1/2}$ space.

The space in the following example is a $*^{*}{}_{\alpha}T_{1/2}$ space but not a T_b space.

Example 4.39: Let X and τ be as in example 3.3. Here (X, τ) is a $^{**}{}_{\alpha}T_{1/2}$ space but not a T_b space. Since a, c} is a gs-closed set but not closed set.

Theorem 4.40: Every $_{\alpha}T_b$ space is a $^{**}{}_{\alpha}T_{1/2}$ space.

Proof: Let A be a g-closed set of (X, τ). Since every g-closed set is αg-closed, A is αg-closed set. Since (X, τ) is an $_{\alpha}T_b$ space, A is closed. Since every closed set is *gα-closed, A is *gα-closed. Therefore (X, τ) is an $^{**}{}_{\alpha}T_{1/2}$ space.

The space in the following example is a $^{**}{}_{\alpha}T_{1/2}$ space but not an $_{\alpha}T_b$ space.

Example 4.41: Let X and τ be as in example 3.3.Here (X, τ) is a $^{**}{}_{\alpha}T_{1/2}$ space but not an $_{\alpha}T_b$ space. Since {a, c} is a αg-closed set but not closed set.

Theorem 4.42: Every T_c^{**} space is a $^{**}{}_{\alpha}T_{1/2}$ space.

Proof: Let A be a g-closed set of (X, τ). Since every g-closed set is gs-closed, A is gs-closed set. Since (X, τ) is a T_c^{**} space, A is *gα-closed. Therefore (X, τ) is an $^{**}{}_{\alpha}T_{1/2}$ space.

The space in the following example is a $^{**}{}_{\alpha}T_{1/2}$ space but not a T_c^{**} space.

Example 4.43: Let X and τ be as in example 4.11. Here (X, τ) is an $^{**}{}_{\alpha}T_{1/2}$ space but not a T_c^{**} space. Since {a} is a gs-closed set but not a *gα-closed set.

Theorem 4.44: The space (X, τ) is a $T_{1/2}$ space if and only if it is a $^{**}{}_{\alpha}T_{1/2}$ space and an $_{\alpha}T_{1/2}{}^{**}$ space.

Proof: Necessity part: By theorem 4.36 and 4.3.

Sufficient part: Let A be a g-closed set of (X, τ). Since (X, τ) is a $^{**}{}_{\alpha}T_{1/2}$ space, A is *gα-closed. Since (X, τ) is an $_{\alpha}T_{1/2}{}^{**}$ space, A is closed. Therefore (X, τ) is an $T_{1/2}$ space.

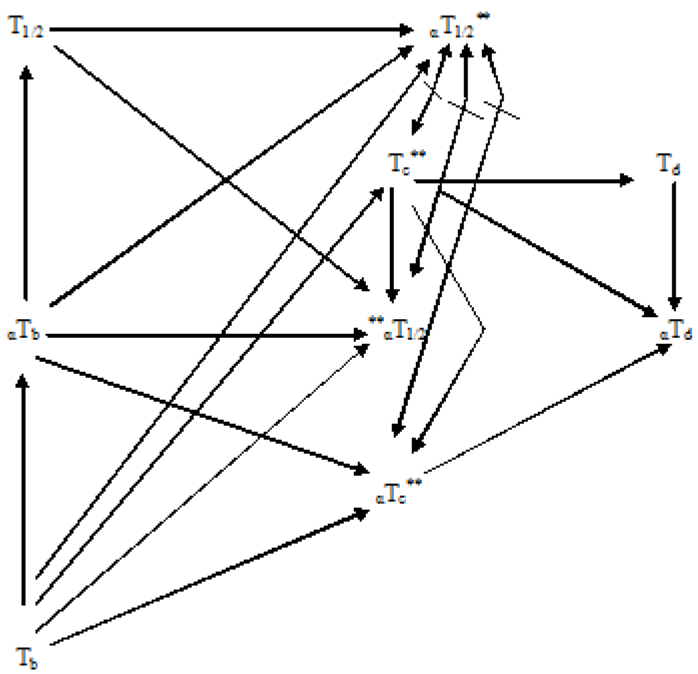
Remark 4.45: $^{**}{}_{\alpha}T_{1/2}$ space and $_{\alpha}T_{1/2}{}^{**}$ space are independent of each other.

It can be seen by the following examples.

Example 4.46: Let X and τ be as in example 3.5. Here (X, τ) is an $_{\alpha}T_{1/2}{}^{**}$ space but not an $^{**}{}_{\alpha}T_{1/2}$ space. Since {b} is g-closed set but not *gα-closed set.

Example 4.47: Let X and τ be as in example 3.3. Here (X, τ) is an $^{**}{}_{\alpha}T_{1/2}$ space but not an $_{\alpha}T_{1/2}{}^{**}$ space. Since {b, c} is *gα **-**closed set but not closed set.

Remark 4.48: The following diagram shows them relationship among the separation axioms considered in this paper and reference [18], [19]. A → B (A↮ B) represents A implies B but B need not imply A always (A and B are independent of each other).



5. *gα – CONTINUITY AND *gα – IRRESOLUTNESS:

We introduce the following definition

Definition 5.1: A function f: (X, τ) →(Y,σ) is called *gα – continuous if $f^{-1}(V)$ is a *gα – closed set of (X, τ) for every closed set V of (Y, σ).

Theorem 5.2: Every continuous map is *gα – continuous.

Proof: Let V be a closed set of (Y,σ). Since f is continuous $f^{-1}(V)$ is closed in (X, τ). But every closed set is *gα-closed set. Hence $f^{-1}(V)$ is *gα-closed set in (X, τ). Thus f is *gα – continuous.

The converse of the above theorem need not be true by the following example.

Example 5.3: Let X= {a, b, c} =Y with τ={X, ϕ, {a, b}} and σ={Y, ϕ, {a}}.

Define f: (X, τ) →(Y, σ) by f(a)=b, f(b)=a, f(c)=c.
*Gα C(X, τ) ={X, ϕ, {c}, {b, c}, {a, c}}.

Here $f^{-1}(\{b, c\}) = \{a, c\}$ is not a closed set in (X, τ). Therefore f is not continuous. However f is $^*g\alpha$ – continuous.

Theorem 5.4: Every $^*g\alpha$ – continuous map is g–continuous.

Proof: Let V be a closed set of (Y,σ).Since f is $^*g\alpha$ – continuous, $f^{-1}(V)$ is $^*g\alpha$ –closed in (X, τ). But every $^*g\alpha$ –closed set is g-closed set. Hence $f^{-1}(V)$ is g-closed set in (X, τ). Thus f is g – continuous.

The converse of the above theorem need not be true by the following example.

Example 5.5: Let X= {a, b, c} =Y with $\tau=\{X, \phi, \{a\},\{a, c\}\}$ and $\sigma=\{Y, \phi, \{a\},\{a, b\}\}$.
Define f: $(X, \tau) \to (Y,\sigma)$ **by** f(a)=b, f(b)=c, f(c)=a.
$^*G\alpha\, C(X, \tau) =\{X, \phi, \{b\},\{b, c\}\}$.
$GC(X, \tau) =\{X,\phi, \{b\}, \{a, b\},\{b, c\}\}$.

Here $f^{-1}(\{b, c\}) = \{a, b\}$ is not a $^*g\alpha$ -closed set in (X, τ). Therefore f is not $^*g\alpha$ -continuous. However f is g– continuous.

Theorem 5.6: Every $^*g\alpha$ – continuous map is gα-continuous.

Proof: Let V be a closed set of (Y,σ). Since f is $^*g\alpha$ – continuous $f^{-1}(V)$ is $^*g\alpha$ –closed in (X, τ). But every $^*g\alpha$ –closed set is gα-closed set in (X, τ). Hence $f^{-1}(V)$ is gα-closed set in (X, τ). Thus f is gα-continuous.

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The converse of the above theorem need not be true by the following example.

Example 5.7: Let X= {a, b, c} =Y with $\tau =\{ X, \phi, \{a\},\{b, c\}\}$ and $\sigma=\{Y, \phi, \{a\},\{a, c\}\}$.
Define f: $(X, \tau) \to (Y,\sigma)$ by f (a) =b, f (b) =c, f(c) =a.
$^*G\alpha\, C(X, \tau) =\{X, \phi, \{a\},\{b, c\}\}$.
$G\alpha C(X, \tau) =\{X,\phi, \{a\},\{b\}, \{c\}, \{a, b\},\{b, c\}, \{a, c\}\}$.

Here $f^{-1}(\{b, c\}) = \{a, b\}$ is not a $^*g\alpha$ -closed set in (X, τ). Therefore f is not $^*g\alpha$ -continuous. However f is gα-continuous.

Remark 5.8: Every $^*g\alpha$ – continuous map is αg-continuous, gs-continuous, gsp-continuous and gpr-continuous.

Theorem 5.9: Every $^*g\alpha$ – continuous map is gp-continuous.

Proof: Let V be a closed set of (Y, σ). Since f is $^*g\alpha$ – continuous $f^{-1}(V)$ is $^*g\alpha$ –closed in (X, τ). But every $^*g\alpha$ –closed set is gp-closed set in (X, τ). Hence $f^{-1}(V)$ is gp-closed set in (X, τ). Thus f is gp-continuous.

The converse of the above theorem need not be true by the following example.

Example 5.10: Let (X, τ) and (Y,σ) be as in example 5.7. Here $f^{-1}(\{b, c\}) = \{a, b\}$ is not a $^*g\alpha$ -closed set in (X, τ). Therefore f is not $^*g\alpha$ -continuous. However f is gp-continuous.

Remark 5.11: $^*g\alpha$ –continuity is independent of semi-continuity and α-continuity.

The proof follows from the following example.

Example 5.12: Let X= {a, b, c} =Y with $\tau =\{X, \phi, \{a\}\}$ and $\sigma=\{Y, \phi, \{a\},\{a, b\}\}$.
Define f: $(X, \tau) \to (Y,\sigma)$ by f (a) =a, f (b) =b, f(c) =c.
$^*G\alpha\, C(X, \tau) =\{X, \phi, \{b, c\}\}$.
$SC(X, \tau) =\{X,\phi, \{b\}, \{c\}, \{b, c\}\}= \alpha C(X, \tau)$

Here $f^{-1}(\{b\}) = \{b\}$ is not a $^*g\alpha$ -closed set in (X, τ). Therefore f is not $^*g\alpha$ -continuous. However f is semi–continuous and α-continuous.

Example 5.13: Let X={a, b, c}=Y with $\tau =\{ X, \phi, \{a, b\}\}$ and $\sigma=\{Y, \phi, \{b\},\{b, c\}\}$.
Define f: $(X, \tau) \to (Y, \sigma)$ by f(a)=c, f(b)=b, f(c)=a.
$^*G\alpha C(X, \tau) =\{X,\phi, \{c\},\{b, c\},\{a, c\}\}$. $SC(X, \tau) =\{X,\phi, \{c\}\}= \alpha C(X, \tau)$

Here $f^{-1}(\{a, c\}) = \{a, c\}$ is not a semi -closed set and α-closed set in (X, τ). Therefore f is not semi-continuous and α-continuous. However f is $^*g\alpha$ -continuous.

Remark 5.14: The composition of two *gα –continuous map need not be a *gα –continuous.

The proof follows from the example.

Example 5.15: Let X={a, b, c}=Y =Z with τ={ X, ϕ, {a},{a, b}} ,σ={Y, ϕ, {a, b}} and η={ Z, ϕ, {b},{b, c}}
Define f: (X, τ) → (Y, σ) by f(a)=a, f(b)=b, f(c)=c.
Define g: (Y, σ) → (Z, η) by g(a)=c, g(b)=b, g(c)=a.
*Gα C(X, τ)={X, ϕ, {c},{b, c}}.
*Gα C(Y, σ)={Y, ϕ, {c},{b, c},{a, c}}.

Clearly f and g are *gα –continuous.

Here {a, c} is a closed set in (Z,η). But $(gof)^{-1}$({a, c}) = {a, c} is not a *gα –closed set in (X, τ).

Therefore gof is not *gα -continuous.

We introduce the following definition.

Definition 5.16: A function f: (X, τ) → (Y, σ) is called *gα –irresolute if $f^{-1}(V)$ is a *gα –closed set of (X, τ) for every *gα –closed set of (Y, σ).

Theorem 5.17: Every *gα –irresolute function is *gα -continuous.

Proof: Let V be a closed set of (Y,σ). Since every closed set is *gα -closed set. Therefore V is *gα -closed set of (Y,σ). Since f is *gα – irresolute $f^{-1}(V)$ is *gα –closed in (X, τ). Therefore f is *gα -continuous.

The converse of the above theorem need not be true by the following example.

Example 5.18: Let X={a, b, c}=Y with τ={ X, ϕ, {b},{b, c}} and σ={Y, ϕ, {a, b}}.
Define f: (X, τ) →(Y, σ) by f(a)=c, f(b)=a, f(c)=b.
*Gα C(X, τ) ={X, ϕ, {a},{b, c}}.
*Gα C(Y, σ) ={Y, ϕ, {c},{b, c},{b, c}}.

Here f is *gα –continuous but f is not *gα –irresolute. Since {a, c} is *gα –closed set in (Y, σ) but f^{-1}({a, c})={a, b} is not *gα –closed set in (X, τ).

Theorem 5.19: Let f: (X, τ) → (Y, σ) and g: (Y, σ) →(Z, η) be any two functions. Then
(i) gof: (X, τ) →(Z, η) is *gα –continuous if g is continuous and f is *gα –continuous.
(ii) gof: (X, τ) → (Z, η) is *gα –irresolute if both g and f are *gα – irresolute.
(iii) gof: (X, τ) → (Z, η) is *gα –continuous if g is *gα –continuous and f is *gα – irresolute.

Proof:
(i) Let V be a closed set in (Z, η). Since g is continuous, $g^{-1}(V)$ is closed in (Y, σ). Since f is *gα –continuous, $f^{-1}(g^{-1}(V))= (gof)^{-1}(V)$ is *gα –closed in (X, τ). Therefore gof is *gα –continuous.

Similarly we can prove (ii) and (iii).

Theorem 5.20: Let f: (X, τ)→ (Y, σ) be a *gα –continuous(resp.gs-continuous, αg-continuous, g-continuous) map. If (X, τ) is an $_\alpha T_{1/2}^{**}$ (resp. T_c^{**}, $_\alpha T_c^{**}$, $^{**}{}_\alpha T_{1/2}$) space, then f is continuous (*gα –continuous, *gα –continuous, *gα –continuous).

Proof: Let V be a closed set of (Y, σ).Since f is *gα –continuous (resp.gs-continuous, αg-continuous, g-continuous), $f^{-1}(V)$ is *gα –closed (resp.gs-closed, αg-closed, g-closed) in (X, τ). Since (X, τ) is an $_\alpha T_{1/2}^{**}$ space(resp. T_c^{**}, $_\alpha T_c^{**}$, $^{**}{}_\alpha T_{1/2}$ space), $f^{-1}(V)$ is closed (*gα –closed) in (X, τ). Therefore f is continuous (*gα –continuous, *gα –continuous, *gα –continuous).

Theorem 5.21: Let f: (X, τ) → (Y, σ) be a surjective, gα-irresolute and a closed map. Then f(A) is *gα –closed set of (Y, σ) for every *gα –closed set A of (X, τ).

Proof: Let A be a *gα –closed set of (X, τ). Let U be a gα-open set of (Y, σ) such that f(A)⊆U. Since f is surjective and gα-irresolute, $f^{-1}(U)$ is a gα-open set of (X, τ). Since A⊆ $f^{-1}(U)$ and A is *gα –closed set of (X, τ),cl(A) ⊆ $f^{-1}(U)$.

Then f(cl(A)) ⊆ f(f^{-1}(U)) = U. Since f is closed, f(cl(A))=cl(f(cl(A))). This implies cl(f(A)) ⊆ cl(f(cl(A)))= f(cl(A)) ⊆U. Therefore f(A) is a *gα –closed set of (Y, σ).

Theorem 5.22: Let f: (X, τ) →(Y,σ) be a surjective, *gα-irresolute and a closed map. If (X, τ) is an $_{\alpha}T_{1/2}^{**}$ space, then (Y,σ) is also an $_{\alpha}T_{1/2}^{**}$ space.

Proof: Let A be a *gα –closed set of (Y,σ). Since f is *gα-irresolute, f^{-1}(A) is a *gα-closed set of (X, τ). Since (X, τ) is an $_{\alpha}T_{1/2}^{**}$ space, f^{-1}(A) is a closed set of (X, τ). Then f(f^{-1}(A)) = A is closed in (Y, σ). Thus A is a closed set of (Y,σ). Therefore (Y,σ) is a $_{\alpha}T_{1/2}^{**}$ space.

Definition 5.23: A function f: (X, τ) →(Y, σ) is called pre-*gα-closed if f(A) is a *gα-closed set of (Y, σ) for every *gα-closed set A of (X, τ).

Theorem 5.24: Let f: (X, τ) → (Y,σ) be a surjective, gs-irresolute and a pre-*gα-closed map. If (X, τ) is an T_c^{**} space, then (Y,σ) is also an T_c^{**} space.

Proof: Let A be a gs –closed set of (Y,σ). Since f is gs-irresolute, f^{-1}(A) is a gs-closed set in (X, τ). Since (X, τ) is a T_c^{**} space, f^{-1}(A) is a *gα-closed set in (X, τ). Since f is pre-*gα-closed map, f(f^{-1}(A)) is *gα-closed in (Y, σ) for every *gα-closed set f^{-1}(A) of (X, τ). Since f is surjection, A=f (f^{-1}(A)). Thus A is a *gα-closed set of (Y,σ). Therefore (Y, σ) is a T_c^{**} space.

Theorem 5.25 Let f: (X, τ) →(Y,σ) be a surjective, αg-irresolute and a pre-*gα-closed map. If (X, τ) is an $_{\alpha}T_c^{**}$ space, then (Y, σ) is also an $_{\alpha}T_c^{**}$ space.

Proof: Let A be a αg–closed set of (Y, σ). Since f is αg -irresolute, f^{-1}(A) is a αg -closed set in (X, τ). Since (X, τ) is a $_{\alpha}T_c^{**}$ space, f^{-1}(A) is a *gα-closed set in (X, τ). Since f is pre-*gα-closed map, f(f^{-1}(A)) is *gα-closed in (Y, σ) for every *gα-closed set f^{-1}(A) of (X, τ). Since f is surjection, A=f(f^{-1}(A)). Thus A is a *gα-closed set of (Y, σ). Therefore (Y,σ) is a $_{\alpha}T_c^{**}$ space.

Theorem 5.26: Let f: (X, τ) → (Y,σ) be a surjective, gc-irresolute and a pre-*gα-closed map. If (X, τ) is an $^{**}{}_{\alpha}T_{1/2}$ space, then (Y, σ) is also an $^{**}{}_{\alpha}T_{1/2}$ space.

Proof: Let A be a g–closed set of (Y,σ). Since f is gc-irresolute, f^{-1}(A) is a g -closed set in(X, τ). Since (X, τ) is a $^{**}{}_{\alpha}T_{1/2}$ space, f^{-1}(A) is a *gα-closed set in (X, τ). Since f is pre-*gα-closed map, f(f^{-1}(A)) is *gα-closed in (Y, σ) for every *gα-closed set f^{-1}(A) of (X, τ). Since f is surjection, A=f(f^{-1}(A)). Thus A is a *gα-closed set of (Y, σ). Therefore (Y, σ) is a $^{**}{}_{\alpha}T_{1/2}$ space.

6.*Generalized αc - homeomorphism and their group structure

Definition 6.1: A function f: (X, τ) →(Y,σ) is said to be *gα-open if the image f (U) is *gα-open in (Y, σ) for every open set U of (X, τ).

Definition 6.2: A function f: (X, τ) → (Y,σ) is said to be *gα-closed if the image f (U) is *gα-closed in (Y,σ) for every closed set U of (X, τ).

Definition 6.3: A function f: (X, τ) →(Y,σ) is said to be *gαc- homeomorphism (resp.*gα- homoeomorphism) if f is bijective and f and f^{-1} are *gα-irresolute (resp. *gα-continuous).

Theorem 6.4:
(i) Suppose that f is bijection. Then the following conditions are equivalent:
(1) f is *gα- homoeomorphism.
(2) f is *gα-open and *gα-continuous.
(3) f is *gα-closed and *gα-continuous.
(ii) If f is a homeomorphism, then f and f^{-1} are *gα-irresolute.
(iii) Every *gαc- homeomorphism is a *gα- homoeomorphism.

Proof:
(ii) First we prove that f^{-1} is *gα-irresolute. Let A be a *gα-closed set of (X, τ). To show $(f^{-1})^{-1}$(A) = f(A) is *gα-closed in (Y, σ). Let U be a gα-open set such that f(A) ⊆ U. Then A = (f^{-1}(f(A)) ⊆ f^{-1}(U) is gα-open. Since A is *gα-closed, cl(A) ⊆ f^{-1}(U).We have cl(f(A)) ⊆ f(cl(A)) ⊆ f(f^{-1}(U)) = U and so f(A) is *gα-closed. Thus f^{-1} is *gα-irresolute. Since f^{-1} is also a homeomorphism $(f^{-1})^{-1}$ = f is *gα-irresolute.

(iii) Let f is bijective. Since f is *gαc- homeomorphism, f and f^{-1} are *gα-irresolute. Since every *gα-irresolute map is *gα-continuous, then f and f^{-1} are *gα-continuous. Therefore f is *gα- homoeomorphism.

Definition 6.5: For a topological space (X, τ) we define the following three collections of functions:
(i) *gαch (X, τ) = {f/ f: (X, τ) → (X, τ) is a *gαc- homeomorphism}.
(ii) *gαh (X, τ) = {f/ f: (X, τ) → (X, τ) is a *gα- homeomorphism}.
(iii) h (X, τ) = {f/ f: (X, τ)→ (X, τ) is a homeomorphism}.

Corollary 6.6: For a space (X, τ) the following properties hold.
(i) h (X, τ) ⊆ *gαch (X, τ) ⊆ *gαh (X, τ).
(ii) The set *gαch (X, τ) forms a group under composition of functions.
(iii) The group h (X, τ) is a subgroup of *gαch (X, τ).
(iv) If f: (X, τ) →(Y,σ) is a *gαc- homeomorphism then it induces an isomorphism f_*: *gαch (X, τ) → *gαch(Y, σ).

Proof:
(i) These implications are obtained by theorem 6.4(ii), (iii).
(ii) By theorem 5.19.
(iii) By (i).
(iv) We define f_*: *gαch (X, τ) → *gαch(Y, σ) by $f_*(h)$ = fohof^{-1}. Then using 5.19 we have that $f_*(h)\in$ *gαch (X, τ). It is shown that f_* is the required group isomorphism.

Remark 6.7: The following example shown that the converse of the above theorem (iv) is not true.

Example 6.8: Let X = {a, b, c}= Y with with τ = { X, ϕ, {a},{b, c}} and σ = {Y, ϕ, {a, b}}.
Define f: (X, τ) →(Y, σ) by f(a)=c, f(b)=a, f(c)=b.
*Gα C(X, τ)={X, ϕ, {a},{b, c}}.
*Gα C(Y, σ) ={Y, ϕ, {c},{b, c},{a,c}}.

Also define three functions h_a, h_b, h_c: (X, τ)→(X, τ) by
$h_a(a)$=a, $h_a(b)$=c, $h_a(c)$=b
$h_b(a)$=a, $h_b(b)$=b, $h_b(c)$=c
$h_c(a)$=b, $h_c(b)$=a, $h_c(c)$=c

Then it is shown that *gαch (X, τ) = {1_x, h_a}, *gαch(Y,σ) = {1_y, h_c} and f_*: *gαch (X, τ) → *gαch(Y,σ) is an isomorphism such that $f_*(h_a) = h_b$. However f is not *gαc- homeomorphism.

7. EXAMPLES IN THE DIGITAL PLANE

In the digital plane, we investigate explicite forms of GαO-kernel α-kerenl and of a subset. The digital line or the so called Khalimsky line is the set of the integers Z, equipped with the topology k having {{2n+1,2n,2n –1}/n∈Z} as a subbase. This is denoted by (Z, k). Thus, a subset U is open in (Z, k) if and only if whenever x∈U is an even integer, then x-1,x+1∈U. Let (Z^2,k^2) be the topological product of two digital lines (Z, k), where Z^2 =Z×Z and k^2=k×k. This space is called the digital plane in the present paper(cf.[5],[11],[12]). We note that for each point x∈ Z^2 there exists the smallest open set containing x, say U(x). For the case of x = (2n+1,2m+1), U(x)={2n+1}× {2m+1}; for the case of x = (2n,2m), U(x) = {2n-1,2n,2n+1}× {2m-1,2m,2m+1}; for the case of x = (2n,2m+1), U(x) = {2n-1,2n,2n+1}× {2m+1}; for the case of x = (2n+1,2m), U(x) = {2n+1}× 2m-1,2m,2m+1}, where n,m∈Z. For a subset E of (Z^2,k^2), we define the following three subsets as follows: E_F = {x∈E/ x is closed in (Z^2,k^2)}; E_{k^2} = {x∈E/ x is open in (Z^2,k^2)}; E_{mix}= E\($E_F \cup E_{k2}$). Then it is shown that E_F = {(2n, 2m)∈E/ n,m ∈ Z},E_{k^2}= {(2n+1, 2m+1)∈E/ n,m∈ Z} and E_{mix}={(2n, 2m+1) ∈E/ n,m ∈ Z}∪ {(2n+1, 2m) ∈E/ n,m ∈ Z}.

Theorem 7.1: Let A and E be subsets of (z^2, k^2).
(i) If E is non - empty α–closed set, then $E_F \neq \phi$[8].
(ii) If E is α– closed and E ⊆ $B_{mix} \cup B_{k^2}$ holds for some subset B of (z^2, k^2) then E = ϕ[8].
(iii) The set U $(A_F) \cup A_{mix} \cup A_{k^2}$ is a gα-open set containing A.

Proof:
(iii): We claim that $A_{mix} \cup A_{k^2}$ is a gα-open set . Let F be any non –empty α– closed set such that F ⊆ $A_{mix} \cup A_{k^2}$. Then by (ii) , F = ϕ. Thus, we have that F ⊆ α - Int $(A_{mix} \cup A_{k^2})$ then $A_{mix} \cup A_{k^2}$ is gα - open . But we know that U (A_F) is a open set. Then U $(A_F) \cup A_{mix} \cup A_{k^2}$ is gα-open by theorem 3.14. But A = $A_F \cup A_{mix} \cup A_{k^2}$. A ⊆ U$(A_F) \cup A_{mix} \cup A_{k^2}$. This implies that gα-open set contains A.

Theorem 7.2: Let A be a subset of (Z^2, k^2). The Gαo– kernel of A and the α-kernel of A are obtained precisely as follows:
(i) Gαo–ker (A) = $U(A_F) \cup A_{mix} \cup A_{k^2}$, where $U(A_F) = \cup\{U(x) \mid x \in A_F\}$.
(ii) α-ker (A) = U(A), where $U(A) = \cup\{U(x) \mid x \in A\}$[8].

Proof:
(i): Let $U_A = U(A_F) \cup A_{mix} \cup A_{k^2}$. By Lemma 7.1 (iii), Gαo - ker (A) $\subseteq U_A$.

To prove $U_A \subseteq$ Gαo-ker (A), it is claimed that (⁎)if there exists a gα-open set V such that $A \subseteq V \subset U_A$ then $V = U_A$. Indeed, let x be any point of U_A. There are three cases for the point x.

Case (1): $x \in (U_A)_F$. we note that $(U_A)_F = (U(A_F))_F \cup (A_{mix} \cup A_{k^2})_F = A_F$.

Then we have that $x \in A_F \subseteq A \subseteq V$.

Case (2): $X \in (U_A)_{k^2}$. We note that

$(U_A)_{k^2} = (U(A_F)_{k^2}) \cup (A_{mix})_{k^2} \cup (A_{k^2})_{k^2} = (U(A_F))_{k^2} \cup A_{k^2}$.

Firstly suppose that $x \in U(A_F)$ Then $x \in U(y)$ for some $y \in A_F$. Since $y \in A_F \subseteq A \subseteq V$ and V is gα-open, we have $\{y\} \subseteq$ α-Int (V). Then $U(y) \subseteq$ α-Int (V), because α-Int (V) is α-open. Thus we have that $x \in V$.

Secondly, suppose $x \in A_{k^2}$, then we have $x \in V$, because $x \in A_{k^2} \subseteq A \subseteq V$.

Case (3): $x \in (U_A)_{mix}$. We note that

$$(U_A)_{mix} = (U(A_F))_{mix} \cup (A_{k^2})_{mix} \cup (A_{mix})_{mix}$$
$$= (U(A_F))_{mix} \cup A_{mix}$$

Firstly suppose that $x \in U(A_F)$. Then $x \in U(y)$ for some $y \in A_F$. Then y be a α–closed point since every closed point is α–closed point. Since $y \in A_F \subseteq A \subseteq V$, {y} is α–closed and V is gα - open set, we have $\{y\} \subseteq$ α - Int (V). Then U (y) $\subseteq$ α - Int (V) and so $x \in V$.

Secondly, suppose that $x \in A_{mix}$. Then $x \in A_{mix} \subseteq A \subseteq V$ implies $x \in V$.

For all cases we assume that $x \in U_A$ then we show that $x \in V$, then $U_A \subseteq V$. But we know that $V \subseteq U_A$. From the above cases we conclude that $V = U_A$. Thus we shown (⁎).

Let Gαo(A) be the family of all gα-open sets containing A. Then, we have that $U_A \subseteq W$ for each W∈ Gαo(A), using (⁎) above and properties that $A \subseteq W \cap U_A \subseteq U_A$ and $W \cap U_A$ is gα -open set. Hence, we show that $U_A \subseteq \cap\{W / W \in$ Gαo(A)}= Gαo–ker (A).

That is $U_A \subseteq$ Gαo–ker (A). Therefore Gαo–ker (A) = U_A.

Theorem 7.3: Let E be a subset of (Z^2, k^2).
(i) If E is a non-empty gα-closed set, then $E_F \neq \phi$.
(ii) If E is gα-closed set and $E \subseteq B_{mix} \cup B_{k^2}$ holds for some subset B of (Z^2, k^2), then E = ϕ.

Proof:
(i): We recall that a subset E is gα-closed if and only if αcl(E) ⊆ α-ker (E).Let y be a point of E.

We consider the following three cases for the point y.

Case 1: $y \in E_{k^2}$. Let y = (2n+1, 2m+1) for some n,m ∈ Z. Then αcl(y)= {2n, 2n+1, 2n+2} × {2m, 2m+1, 2m+2}⊆αcl(E) ⊆ α-ker (E). Thus there exists a point (2n, 2m)∈ α-ker (E), say y_1 = (2n, 2m). Using theorem 7.2(ii), we have that $y_1 \in U(z)$ for some z∈E.

If $z \in E_{mix}$, say z = (2s+1, 2t) for some s,t∈Z, then U(z) = {2s+1}× {2t-1,2t,2t+1} and $y_1 \notin U(Z)$. This is a contradiction.

Next if $z \in E_k^{2}$, say z = (2s+1, 2t+1) for some $s,t \in Z$, then U(z) = {(2s+1, 2t+1)} and $y_1 \notin U(z)$. This is also a contradiction.

Thus we have that $z \in E_F$ and hence $E_F \neq \phi$ for case1.

Case 2: $y \in E_{mix}$. Let y = (2n+1, 2m) for some $n,m \in Z$. Then $\alpha cl(y) = \{2n, 2n+1, 2n+2\} \times \{2m\} \subseteq \alpha cl(E) \subseteq \alpha$-ker (E). Thus there exists a point $(2n, 2m) \in \alpha$-ker (E), say $y_1 = (2n, 2m)$. Using theorem 7.2(ii), we have that $y_1 \in U(z)$ for some $z \in E$.

If $z \in E_{mix}$, say z = (2s+1, 2t) for some $s,t \in Z$, then $U(z) = \{2s+1\} \times \{2t-1, 2t, 2t+1\}$ and $y_1 \notin U(z)$. This is a contradiction.

Next if $z \in E_k^{2}$, say z = (2s+1, 2t+1) for some $s,t \in Z$, then U(z) = {(2s+1, 2t+1)} and $y_1 \notin U(z)$. This is also a contradiction.

Thus we have that $z \in E_F$ and hence $E_F \neq \phi$ for case2.

Case 3: $y \in E_F$. Then $E_F \neq \phi$.

We shown that $E_F \neq \phi$ for all cases.

(ii): Suppose that $E \neq \phi$. By (i) we have that $E_F \neq \phi$. It follows from assumption and definition that $E_F \subseteq (B_{mix} \cup B_k^{2})_F = \phi$. We have a contradiction.

Theorem 7.4: Let A be a subset in (Z^2, k^2).
(i) If $(Z^2)_F \subseteq A$ holds, then A is $^*g\alpha$-closed.
(ii) If $(Z^2)_F \subseteq A$ holds and there exists a point $x \in A_k^{2}$ such that $cl\{x\} \subseteq A$, then A is $^*g\alpha$-closed set which is not α-closed.

Proof:
(i) Using theorem 7.2, we have Gαo-ker (A) = $U(A_F) = Z^2$. Then, A is $^*g\alpha$-closed set by theorem 3.21.
(ii) By(i), A is $^*g\alpha$-closed set. Since $\{x\} \subseteq A_k^{2} \subseteq A$ and $Int(cl(\{x\})) = \{x\}$, we have that $cl(\{x\}) \subseteq cl(Int(cl(A)))$ and so $cl(\{x\}) \subseteq \alpha cl(A)$. Suppose that A is α-closed. Then, we have that $cl(\{x\}) \subseteq A$. This is a contradiction.

Example 7.5: The converse of the theorem 7.3(i) is not true in general. A set A={x, y, z} where x = (3, 3), y = (3, 2) and z = (4, 2) is not gα-closed but $A_F \neq \phi$.

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