

GENERALIZED FIBONACCI-LUCAS SEQUENCE

Mohsen Maqbool Dar\*

PG Department of Mathematics, Govt. College Baramulla, Jammu & Kashmir, India

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ABSTRACT

In this paper, we study Generalized Fibonacci-Lucas sequence  $\{H_n\}$  defined by the recurrence relation

$$H_n = H_{n-1} + H_{n-2}, \text{ for all } n \geq 2$$

$H_0 = 2$  and  $H_1 = m+1$ ,  $m$  being a fixed positive integer. The associated initial conditions are the sum of initial conditions of Lucas sequence and  $m$  times the initial conditions of Fibonacci sequence respectively. We shall define Binet's formula and generating function of Generalized Fibonacci-Lucas sequence.

Mainly, Induction method and Binet's formula will be used to establish properties of Generalized Fibonacci-Lucas sequence.

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**Keywords:** Fibonacci Sequence, Lucas Sequence, Generalized Fibonacci-Lucas Sequence.

1. INTRODUCTION

The generalization of Fibonacci and Lucas Sequences leads to several nice and interesting sequences [3] [10].

The sequence of Fibonacci numbers  $\{F_n\}$  is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \quad F_0 = 0, F_1 = 1. \tag{1.1}$$

The sequence of Lucas numbers  $\{L_n\}$  is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \quad L_0=2, \quad L_1=1. \tag{1.2}$$

The Binet's formula for Fibonacci sequence is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}} \left\{ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right\} \tag{1.3}$$

where  $\alpha = \frac{1+\sqrt{5}}{2} = \text{Golden ratio} \approx 1.618$

and  $\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$ .

Similarly, the Binet's formula for Lucas sequence is given by

$$L_n = \alpha^n + \beta^n = \left\{ \left( \frac{1+\sqrt{5}}{2} \right)^n + \left( \frac{1-\sqrt{5}}{2} \right)^n \right\}$$

\*Corresponding author: Mohsen Maqbool Dar\*

PG Department of Mathematics, Govt. College Baramulla, Jammu & Kashmir, India

In this paper, we present various properties of the Generalized Fibonacci-Lucas sequence  $\{H_n\}$  defined by

$$H_n = H_{n-1} + H_{n-2}, \text{ for all } n \geq 2 \tag{1.4}$$

with  $H_0 = 2$  and  $H_1 = m+1$ ,  $m$  being a fixed positive integer.

Here the initial conditions  $H_0$  and  $H_1$  are the sum of initial conditions of Lucas sequence and  $m$  times the initial conditions of Fibonacci sequence respectively.

i.e.  $H_0 = L_0 + mF_0, H_1 = L_1 + mF_1.$

The few terms of the sequence  $\{H_n\}$  are

$$2, m+1, m+3, 2m+4, 3m+7, \text{ and so on.}$$

## 2. PRELIMINARY RESULTS OF GENERALIZED FIBONACCI-LUCAS SEQUENCE

We need to introduce some basic results of Generalized Fibonacci-Lucas sequence and Fibonacci Sequence.

The relation between Fibonacci Sequence and Generalized Fibonacci-Lucas sequence can be written as

$$H_n = L_n + mF_n, \quad n \geq 0. \tag{2.1}$$

The recurrence relation (1.1) has the characteristic equation  $x^2 - x - 1 = 0$ , which has two roots

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}$$

Now notice a few things about  $\alpha$  and  $\beta$ :

$$\alpha + \beta = 1, \quad \alpha - \beta = \sqrt{5} \text{ and } \alpha\beta = -1.$$

using these two roots, we obtain Binet's formula of recurrence relation (1.4)

$$H_n = (\alpha^n + \beta^n) + m \left( \frac{\alpha^n - \beta^n}{\sqrt{5}} \right)$$

$$= \left\{ \left( \frac{1+\sqrt{5}}{2} \right)^n + \left( \frac{1-\sqrt{5}}{2} \right)^n \right\} + \frac{m}{\sqrt{5}} \left\{ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right\}$$

The generating function of  $\{H_n\}$  is defined as

$$\sum_{n=0}^{\infty} H_n x^n = \frac{2 + (m-1)x}{1 - x - x^2}$$

Using partial fractions, we obtain

$$\sum_{n=0}^{\infty} H_n x^n = \frac{1}{2\sqrt{5}} \sum_{n=0}^{\infty} \left[ \frac{(-1)^n a_1}{a^{n+1}} + \frac{b_1}{b^{n+1}} \right] x^n$$

where  $a = \frac{1+\sqrt{5}}{2}, a_1 = (5-m) + \sqrt{5} (m-1)$

$$b = \frac{-1+\sqrt{5}}{2}, b_1 = (5-m) - \sqrt{5} (m-1)$$

### 3. PROPERTIES OF GENERALIZED FIBONACCI-LUCAS SEQUENCE

Despite its simple appearance the Generalized Fibonacci-Lucas sequence  $\{H_n\}$  contains a wealth of subtle and fascinating properties [5].

#### Sums of Generalized Fibonacci-Lucas terms:

**Theorem 3.1:** Sum of first  $n$  terms of the Generalized Fibonacci-Lucas sequence  $\{H_n\}$  is

$$H_1 + H_2 + H_3 + \dots + H_n = \sum_{k=1}^n H_k = H_{n+2} - (m + 3) \tag{3.1}$$

This identity becomes

$$H_1 + H_2 + \dots + H_{2n} = \sum_{k=1}^{2n} H_k = H_{2n+2} - (m + 3) \tag{3.2}$$

**Theorem 3.2:** Sum of the first  $n$  terms with odd indices is

$$H_1 + H_3 + H_5 + \dots + H_{2n-1} = \sum_{k=1}^n H_{2k-1} = H_{2n} - 2 \tag{3.3}$$

**Theorem 3.3:** Sum of the first  $n$  terms with even indices is

$$H_2 + H_4 + H_6 + \dots + H_{2n} = \sum_{k=1}^n H_{2k} = H_{2n+1} - (m + 1) \tag{3.4}$$

The identities from (3.1) to (3.4) can be derived by induction method.

If we subtract equation (3.4) termwise from equation (3.3), we get alternating sum of first  $n$  numbers

$$\begin{aligned} H_1 - H_2 + H_3 - H_4 + \dots + H_{2n-1} - H_{2n} &= H_{2n} - 2 - H_{2n+1} + m + 1 \\ &= - H_{2n-1} + m - 1 \end{aligned} \tag{3.5}$$

Adding  $H_{2n+1}$  to both sides of equation (3.5), we get

$$\begin{aligned} H_1 - H_2 + H_3 - H_4 + \dots + H_{2n-1} - H_{2n} + H_{2n+1} &= - H_{2n-1} + m - 1 + H_{2n+1} \\ &= H_{2n} + m - 1 \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$H_1 - H_2 + H_3 - H_4 + \dots + (-1)^{n+1} H_n = (-1)^{n+1} H_{n-1} + m - 1 \tag{3.7}$$

**Theorem 3.4:** Sum of the squares of first  $n$  terms of the Generalized Fibonacci-Lucas Sequence is

$$H_1^2 + H_2^2 + H_3^2 + \dots + H_n^2 = \sum_{k=1}^n H_k^2 = H_n H_{n+1} - 2 (m + 1) \tag{3.8}$$

Now we state and prove some nice identities similar to those obtained for Fibonacci and Lucas sequences [6]

**Theorem 3.5:** For every integer  $n \geq 0$ ,

$$mH_{n+2} - m H_{n+1} = m H_n \tag{3.9}$$

**Theorem 3.6:** For every positive integer  $n$ ,

$$H_n^2 = H_n H_{n+1} - H_{n-1} H_n, \quad n \geq 1 \tag{3.10}$$

**Theorem 3.7:** For every positive integer  $n$ ,

$$H_{n+1} H_{n-1} - H_n^2 = (-1)^n (m^2 - 5) \tag{3.11}$$

**Proof:** we shall use mathematical induction over n.

It is easy to see that for  $n = 1$ ,

$$H_0 H_1 - H_1^2 = (-1)^1 (m^2 - 5) - (m^2 - 5) = (-1)^1 m^2 - 5, \text{ which is true.}$$

Assume that the result is true for  $n = k$ . Then

$$H_{k+1} H_{k-1} - H_k^2 = (-1)^k (m^2 - 5) \tag{3.12}$$

Adding  $H_k H_{k+1}$  to each side of equation (3.12), we get

$$\begin{aligned} H_{k+1} H_{k-1} - H_k^2 + H_k H_{k+1} &= (-1)^k (m^2 - 5) + H_k H_{k+1} \\ H_{k+1} (H_{k-1} + H_k) - H_k^2 &= (-1)^k (m^2 - 5) + H_k H_{k+1} \\ H_{k+1}^2 - H_k H_{k+2} &= (-1)^k (m^2 - 5) \\ H_k H_{k+2} - H_{k+1}^2 &= (-1)^{k+1} (m^2 - 5) \end{aligned}$$

Which is precisely our identity when  $n = k + 1$ .

Therefore, the result is true for  $n = k+1$  also.

Hence,  $H_{n+1} H_{n-1} - H_n^2 = (-1)^n (m^2 - 5) \forall n \geq 1$ .

**Theorem 3.8:** Let n be a positive integer. Then

$$H_{2n} = \sum_{k=0}^n \binom{n}{k} H_{n-k} \tag{3.13}$$

#### 4. CONNECTION FORMULAE

**Theorem 4.1:** Let n be a positive integer. Then

$$H_{n+1} + H_{n-1} = (m+1)L_n + 2L_{n-1}, \quad n \geq 1 \tag{4.1}$$

**Theorem 4.2:** Let n be a positive integer. Then

$$H_{n+1} - H_{n-1} = (m+1)F_n + 2F_{n-1}, \quad n \geq 1 \tag{4.2}$$

**Theorem 4.3:** For every integer  $n \geq 0$ ,

$$H_{n+1} = L_{n+1} + mF_{n+1}, \quad n \geq 0 \tag{4.3}$$

**Theorem 4.4:** For every integer  $n \geq 0$ ,

$$H_{2n} = L_{2n} + mF_{2n}, \quad n \geq 0 \tag{4.4}$$

#### 5. CONCLUSION

There are many known identities established for Fibonacci and Lucas sequences. This paper describes comparable identities of Generalized Fibonacci-Lucas sequence. We have also developed connection formulas for Generalized Fibonacci-Lucas sequence, Fibonacci sequence and Lucas sequence respectively. It is easy to discover new identities simply by varying the pattern of known identities and using inductive reasoning to guess new results. Of course, the ideas can be extended to more general recurrent sequences in obvious way.

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