International Journal of Mathematical Archive-3(6), 2012, 2346-2352
IJMA Available online through www.ijma.info ISSN 2229 – 5046

# Nash-Equilibrium Solutions For Rough Continuous Static Games (N-R CSG)

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*(Received on: 03-06-12; Revised & Accepted on: 20-06-12)*

**ABSTRACT**

*This paper presents a new framework to hybridize the rough set theory with the continuous static games and called "Rough Continuous Static Games". This paper proposes a new formulation, classification and definition of the rough continuous static games (RCSG) problems.*

*In this game, each player knows the cost functions and constraints for all other players and in which no cooperation is possible, so the Nash equilibrium solution concept is suited for this case [3]. Finally a numerical example will be introduced to illustrative the proposed method.*

***Keywords:*** *Rough set, rough optimality, rough feasibility, continuous static games, Nash equilibrium solution concept.*

## INTRODUCTION

For mathematical programming problems (MPPs) in the crisp form, the aim is to maximize or minimize an objective function over feasible set. But in many practical situations, the decision maker may not be in a position to specify the objective and/ or the feasible set precisely but rather can specify them in a "rough sense". In such situations, it is desirable to use some rough programming type of modeling so as to provide more flexibility to the decision maker.

Rough set theory (RST) proposed by Pawlak in 1980 [7, 8], presents still another attempt to deal with vagueness or uncertainty. Uncertain programming was defined as the optimization theory in generally uncertain environment, stochastic programming, fuzzy programming and rough programming is all subtopics of uncertain programming [4, 5].

Rough set theory expresses vagueness employing a boundary region of a set. If the boundary region of set is empty it means that the set is crisp otherwise the set is rough (inexact).

The available information in this theory is represented by an equivalence relation. Associate with every set a pair of classical sets, which are called the lower approximation and the upper approximation of the set. The lower approximation consists of all objects that surely belong to the set of interest, where the upper approximation consists of all objects which possibly belong to the set [2, 6].

We consider the more general case of multiple decision makers, each with their own cost criterion. This generalization introduces the possibility of competition among the system controllers, called "players" and the optimization problem under consideration is therefore termed a "game". Each player in the game controls a specified subset of the system parameters (called his control vector) and seek to minimize his own scalar cost criterion, subject to specified constraints [3].

This paper introduced a new characterization and classification of rough continuous static games problems. New definitions concerning rough optimal sets, rough optimal value, rough optimality and rough feasibility were proposed.

## ROUGH SET AND APPROXIMATION SPACE

Rough set theory has been proven to be an excellent mathematical tool dealing with vague description of objects [7, 8]. A fundamental assumption in rough set theory is that any object from a universe is perceived through available information, and such information may not be sufficient to characterize the object exactly. Pawlak has proposed rough set methodology as a new approach in handling classificatory analysis of vague concept [9]. In this methodology any vague concept is characterized by a pair of precise concepts called the lower and the upper approximations. Rough set theory is based on equivalence relations describing partitions made of classes of indiscernible objects.

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Let $U$ be a non–empty finite set of objects, called the universe, and $R \subseteq U \times U$ be an equivalence relation on $U$. The ordered pair $A = (U, R)$ is called an approximation space generated by $R$ on $U$, the equivalence relation R generates a partition $U / R = \{y_1, y_2, ..., y_m\}$ where $y_1, y_2, ..., y_m$ are the equivalence classes (also called elementary sets) of the approximation space A. In rough set theory, any subset $M \subseteq U$ is described by the elementary sets of A, and the two sets.

$R_*(M) = \bigcup \{y_i \in U / R \,|\, y_i \subseteq M\}$ and $R^*(M) = \bigcup \{y_i \in U / R \,|\, y_i \bigcap M \neq \varphi\}$ are called the lower and upper approximations of $M$, respectively. Therefore, $R_*(M) \subseteq M \subseteq R^*(M)$. The difference between the upper and lower approximations is called the boundary of $M$ and is denoted by $BN_R(M) = R^*(M) - R_*(M)$.

The set $M$ is called exact in A iff $BN_R(M) = \varphi$, otherwise the set $M$ is in exact (rough) in A.

**CLASSES OF ROUGH CONTINUOUS STATIC GAMES (RCSG)**

Continuous Static Games takes the form [3].

Each player, $i = 1, ..., r,$ select his control vector $u^i \in E^{si}$ seeking to minimize a scalar valued criterion

$$G_i(x, u) \tag{1.1}$$

S.T.

$$g(x, u) = 0 \tag{1.2}$$

Where $x \in E^n$ is the state and $u = (u^1, u^2, ... u^r) \in E^s$, $S = S_1 + S_2 + ... + S_r$ the composite control.

The composite control is required to be an element of a regular control constraint set $\Omega \subseteq E^s$ of the form:

$$\Omega = \{ u \in E^s \,|\, h\,(x, u) \geq o \} \tag{1.3}$$

Where $x = \zeta(u)$ is the solution of (1.2) given *u*.

The function $G_i(x, u) : E^n \text{x}\, E^s \to E^1$, $g\,(x, u) : E^n \text{x}\, E^s \to E^n$, and $h\,(x, u) : E^n x\, E^s \to E^q$ are assumed to be $c^1$, with

$$\left| \frac{\partial\, g\,(x, u)}{\partial\, x} \right| \neq 0 \tag{1.4}$$

in a ball about a solution point $(x, u)$. The above problem can be written as:

$$\min\ G_i(x, u)$$

S.T

$$M = \{x \in E^n, u \in E^s \,|\, g\,(x, u) = 0, h\,(x, u) \geq 0\}.$$

Where $G_i$ is called the cost function for each player $i = 1, ..., r,$ and $M$ is called the feasible set of the problem.

In the above formulation, it is assumed that all entries of $G_i(x, u), i = 1, ..., r$ and M are defined in the crisp sense. However, in many practical situations it may not be reasonable to require that the feasible set or the cost function be specific in precise crisp terms. In such situations, it is desirable to use some type of rough modeling and this leads to the concept of rough continuous static games (RCSG).

 

Therefore, the rough continuous static games problems can be classified as [1]:

**1st class:** continuous static games with rough feasible set and crisp cost functions,
**2nd class:** continuous static games with crisp feasible set and rough cost functions,
**3rd class:** continuous static games with rough feasible set and rough cost functions,
**4th class:** continuous static games with crisp feasible set and rough cost functions and crisp for other cost function.

## THE 1ST CLASS OF (RCSG) PROBLEM

Suppose that $A=(U,R)$ is an approximation space generated by an equivalence relation $R$ on the universe $U$, and $U/R=\{y_1, y_2, y_3, ..., y_m\}$ is the partation generated by $R$ on $U$. A rough continuous static games of 1st class problem takes the following form:

$$\begin{aligned} &\min G_i(x,u) \\ &\text{S.T} \\ &M_* \subseteq M \subseteq M^* \end{aligned} \tag{1.5}$$

Where $M \subseteq U$ is a rough set in the approximation space $A=(U,R)$ representing the feasible region of the problem. The sets $M^*$ and $M_*$ represents the notion of "rough–feasibility" of problem (1.5), where $M^*$ is called the set of all possibly–feasible solutions and $M_*$ is called the set of all surely–feasible solutions.

On the other hand $U-M^*$ is called the set of all surely–not feasible solutions. The functions $G_i(x,u), i=1,...,r$ is a crisp real cost function which is continuous on $M^*$.

There exists several solution concepts to solve (RCSG) problem as:

Nash equilibrium concept, the min–max concept, pareto–minimal concept, and Stackeloberg Leader–Follower concept.

In this paper we use the Nash equilibrium concept to solve this problem.

## NASH EQUILIBRIUM SOLUTIONS TO SOLVE (RCSG) PROBLEM

The Nash equilibrium solution concept for situation in which coalitions among players are not possible. It is assumed that the player act independently, without collaboration with any of the other players, and that each player seeks to minimize his own cost function. The information available to each player consists of the cost functions and constraints for each player [3].

The solutions have different degrees of feasibility (surely–feasible, possible–feasible, and surely–not feasible). On the other hand, the solutions have different degree of optimality (surely–Nash optimal, possibly–Nash optimal, and surely–not Nash optimal). As a result of these new concepts, the Nash optimal value of the cost functions and the optimal set of the problem are defined in rough sense.

**Definition 1:** In (RCSG) problem, the Nash optimal values of the cost function $G_i, i=1,2,...,r$ is a rough real number $\overline{G}_i, i=1,...,r$ that is determined roughly by lower and upper bounds denoted $\overline{G}_{i*}$ and $\overline{G}_i^*, i=1,...,r$ respectively [1].

**Remark 1:** If $\overline{G}_{i*}=\overline{G}_i^*$ for any player *i* then the optimal value $\overline{G}_i$ is exact, otherwise $\overline{G}_i$ is rough [1].

Also, the single optimal set of the crisp continuous static programming problem is replaced by four optimal sets covering all possible degree of feasibility and optimality.

**Remark 2:**

1- The set of all surely–feasible, surely–Nash solutions is denoted by $FO_{S\,(SN)}$.

2- The set of all surely–feasible, possibly–Nash solutions denoted by $FO_{S\,(PN)}$.

3- The set of all possibly–feasible, surely–Nash solutions is denoted by $FO_{P\,(SN)}$.

4- The set of all possibly–feasible, possibly–Nash solutions is denoted by $FO_{P\,(PN)}$.

| | | Optimality | |
|---|---|---|---|
| | | Possibly | Surly |
| Feasibility | Possibly | $FO_{P(PN)}$ | $FO_{P(SN)}$ |
| | surely | $FO_{S(PN)}$ | $FO_{S(SN)}$ |

**Proposition 1:**

1- $FO_{S\,(SN)} \subseteq FO_{S\,(PN)} \subseteq FO_{P\,(PN)}$.

2- $FO_{S\,(SN)} \subseteq FO_{P\,(SN)} \subseteq FO_{P\,(PN)}$.

3- $FO_{S\,(SN)} = FO_{S\,(PN)} \bigcap FO_{P\,(SN)}$.

In problem (1.5) the lower and upper bounds of the optimal cost value $\overline{G}_i$ are given by:

$$\overline{G}_{i*} = \inf \left\{ a_i, b_i \right\}$$

$$\overline{G}_i^{\,*} = \inf \left\{ a_i, c_i \right\}$$

$$a_i = \min_{(x,u)\in M_*} G_i(x,u), i = 1,\dots,r$$

$$b_i = \inf \bigcup_{\substack{y\in U|E \\ y\subseteq M_{BN}}} \left\{ \max_{(x,u)\in y} G_i(x,u) \right\}, i = 1,\dots,r$$

$$c_i = \min_{(x,u)\in M_{\beta N}} G_i(x,u), i = 1,\dots,r$$

**Definition 2:** A solution $(x,u)\in M^*$ is surely – Nash solution of (1.5) if and only if $G_i(x,u) = \overline{G}_i^{\,*}$.

**Definition 3:** A solution $(x,u)\in M^*$ is possibly – Nash solution of (1.5) if and only if $G_i(x,u) \le \overline{G}_{i\,*}$.

**Definition 4:** A solution $(x,u)\in M^*$ is surly not –Nash solution of (1.5) if and only if $G_i(x,u) > \overline{G}_{i*}$

**Definition 5:** In the 1st class of (RCSG), the Nash sets are defined as follows:

$$FO_{S\,(SN)} = \left\{ (x,u)\in M_* \mid G_i(x,u) = \overline{G}_i^{\,*}, i = 1,\dots,r \right\}$$

$$FO_{S\,(PN)} = \left\{ (x,u)\in M_* \mid G_i(x,u) \le \overline{G}_{i\,*}, i = 1,\dots,r \right\}$$

$$FO_{P\,(SN)} = \left\{ (x,u)\in M^* \mid G_i(x,u) = \overline{G}_i^{\,*}, i = 1,\dots,r \right\}$$

$$FO_{P\,(PN)} = \left\{ (x,u)\in M^* \mid G_i(x,u) \le \overline{G}_{i\,*}, i = 1,\dots,r \right\}$$

**Theorem 1:** If $\hat{u}_1, \hat{u}_2$ are completely regular local surely Nash solution and possibility Nash minimal solution respectively for the Game (1.5), and $\hat{x} = \zeta(\hat{u}_k)$ is the solution to $g(x,\hat{u}_k) = 0, k = 1,2$ then for each $i = 1,\dots,r$, there exists a vector $\lambda(i)\in E^n$ and a vector $\mu(i)\in E^q$ such that [3].

$$\frac{\partial L_i\left[\hat{x},\hat{u}_k,\lambda(i),\mu(i)\right]}{\partial x}=0$$
$$\frac{\partial L_i\left[\hat{x},\hat{u}_k,\lambda(i),\mu(i)\right]}{\partial u^i}=0$$

$$g\left(\hat{x},\hat{u}_k\right)=0$$
$$\mu^T(i)h\left(\hat{x},\hat{u}_k\right)=0$$
$$h\left(\hat{x},\hat{u}_k\right)\geq 0$$
$$\mu(i)\geq 0$$

Where

$$L_i\left[x,u,\lambda(i),\mu(i)\right]=G_i(x,u)-\lambda^T(i)g(x,u)-\mu^T(i)h(x,u).,\ k=1,2$$

**Example:** Let $U$ be a universal set defined as $U = \{u = (u_1, u_2)\in R^2 | u_1^2+u_2^2\leq 9\}$ and let $k$ be a polytope generated by the following closed half planes

$$h_1=2-x_1-x_2\geq 0\quad,\qquad h_2=2+x_1-x_2\geq 0$$
$$h_3=x_2-x_1+2\geq 0\quad,\qquad h_4=x_1+x_2+2\geq 0$$

Suppose that $R$ is an equivalence relation on $\bigcup$ such that: $U/R = \{R_1, R_2, R_3\}$,

$$R_1=\{(u_1,u_2)\in U=(u_1,u_2)\text{is an interior point of polytope k}\}$$
$$R_2=\{(u_1,u_2)\in U=(u_1,u_2)\text{is a boundary point of polytope k}\}$$
$$R_3=\{(u_1,u_2)\in U=(u_1,u_2)\text{is an exterior point of polytope k}\}$$

Consider the following 1st class RCSG:

$$\min F_1(u_1,u_2)=(u_1-2.5)^2+u_2^2$$
$$\min F_2(u_1,u_2)=-u_1-u_2$$
$$S.T.$$
$$M_*=R_1\cup R_2\quad,\quad M^*=R_1\cup R_2\cup R_3$$

Where player (1) selects $u_1\in R^1$ to minimize $F_1(x_1,u_2)$ and player (2) selects $u_2\in R^1$ to minimize $F_2(u_1,u_2)$. Also, M is a rough feasible region in the approximation space A (U, R) and $M_*, M^*$ are the lower and upper approximations of M; respectively, and the boundary region of M is given by $M_{BN}=R_3$.

**Solution: 1st step:** finding the rough minimal value $\overline{F}=F_{\min}$, where $F = (F_1, F_2)$

$$\overline{F_*}=\inf\{a,b\}$$
$$a=\min_{(u_1,u_2)\in M_*}F(u_1,u_2)$$

By using theorem (1) to find $a$, $b$ and $c$. We get

$$a_1=F_1(2,0)=0.25$$
$$,\qquad a_2=F_2(2,0)=-2$$

$$\because \sup_{(u_1,u_2)\in R_3} F_1(u_1,u_2) = F_1(-3,0) = -30.25$$

$$\sup_{(u_1,u_2)\in R_3} F_2(u_1,u_2) = F_2(-3,0) = -3$$

$$\therefore b = \inf \bigcup_{\substack{y\in U/R \\ y\subseteq M_{BN}}} \left\{ \max_{(u_1,u_2)\in y} F(u_1,u_2) \right\} = \max_{(x_1,u_2)\in R_3} F(u_1,u_2)$$

$$\therefore b_1 = F_1(-3,0) = -30.25$$

$$b_2 = F_2(-3,0) = -3$$

$$\therefore \overline{F_{1^*}} = \inf\{a_1, b_1\} = -30.25$$

$$\overline{F_{2^*}} = \inf\{a_2, b_2\} = -3$$

$$\overline{F}^* = \inf\{a, c\}$$

$$\because \inf_{(u_1,u_2)\in R_3} F_1(u_1,u_2) = F_1(2.5,0) = 0$$

$$\inf_{(u_1,u_2)\in R_3} F_2(u_1,u_2) = F_2(2.5,0) = -2.5$$

$$\therefore C = \min_{(u_1,u_2)\in R_3} F(u_1,u_2) = \min_{(u_1,u_2)\in R_3} F(u_1,u_2) = F(2.5,0)$$

$$C_1 = F_1(2.5,0) = 0$$

$$C_2 = F_2(2.5,0) = -2.5$$

$$\therefore \overline{F}_1^* = \inf\{a_1, c_1\} = 0$$

$$\overline{F}_2^* = \inf\{a_2, c_2\} = -2.5$$

$$\therefore \overline{F}_{1^*} = \inf\{a_1, b_1\} = -30.25$$

$$\overline{F}_{2^*} = \inf\{a_2, b_2\} = -3$$

$$\therefore \overline{F}_1 \in [-30.25, 0]$$

$$\overline{F}_2 \in [-3, -2.5]$$

**2nd step:** finding the rough minimal sets

$$FO_{S(SN)} = \left\{ (u_1^*, u_2^*) \in R_1 \cup R_2 : F_1(u_1^*, u_2^*) = 0, F_2(u_1^*, u_2^*) = -2.5 \right\} = \varphi$$

$$FO_{S(PN)} = \left\{ (u_1^*, u_2^*) \in R_1 \cup R_2 : F_1(u_1^*, u_2^*) \le -30.25, F_2(u_1^*, u_2^*) = -3 \right\} = \{(-3.0)\}$$

$$FO_{P(SN)} = \left\{ (u_1^*, u_2^*) \in R_1 \cup R_2 \cup R_3 : F_1(u_1^*, u_2^*) = 0, F_2(u_1^*, u_2^*) = -2.5 \right\} = \{(2.5,0)\}$$

$$FO_{P(PN)} = \left\{ (u_1^*, u_2^*) \in R_1 \cup R_2 \cup R_3 : F_1(u_1^*, u_2^*) \le -30.25, F_2(u_1^*, u_2^*) \le -3 \right\}$$

$$= \left\{ (u_1, u_2) : (u_1 - 2.5)^2 + x_2^2 \le -30.25, -u_1 - u_2 \le -3 \right\}$$

## CONCLUSION

This paper proposes anew formulation, classification and definition of the rough continuous static games by using Nash Equilibrium solution. Only the 1st class of RCSG problem is defined and its optimal sets are characterized in this paper.

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**Source of support: Nil, Conflict of interest: None Declared**