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A NOTE ON SĂLĂGEAN CARLSON-SHAFFER OPERATOR

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ABSTRACT

In the present work, using Sălăgean and Carlson-Shafer operator we introduce a linear operator SL_λ. The objective is to define the classes $VS_\lambda^\alpha(a,c,n,\beta)$ and $VS_\lambda^\alpha(a,c,n)$ using the above linear operator and for functions belonging to these classes we obtain coefficient estimates and many more properties like extreme points, integral means, unified radii results etc.

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1. INTRODUCTION:

Let $U=\{z\in C:|z|<1\}$ be the open unit disk and A denote the class of functions normalized by

$$f(z)=z+\sum_{m=2}^{\infty}a_m z^m \tag{1.1}$$

which are analytic in the open unit disk U satisfying the conditions $f(0)=f'(0)-1=0.$

The class A is closed under the convolution or Hadamard product

$$(f*g)(z)=z+\sum_{m=2}^{\infty}a_m b_m z^m, \qquad z\in U, \tag{1.2}$$

where f is given by (1.1) and $g(z)=z+\sum_{m=2}^{\infty}b_m z^m$.

For $n\in N_0$, $\lambda\ge 0$, $a,c\in R\setminus Z$, we introduce a linear operator $SL_\lambda : A\to A$ defined by

$$SL_\lambda f(z)=(1-\lambda)[(k*k*\cdots*k)*f](z)+\lambda[\phi(a,c)*f](z), \quad z\in U, \tag{1.3}$$

where $k(z)=z(1-z)^{-2}$ is Koebe function and

$$\phi(a,c;z)=\sum_{m=2}^{\infty}\frac{(a)_{m-1}}{(c)_{m-1}}z^m, \quad |z|<1,\ a,c\neq 0,-1,-2\cdots,$$

is the incomplete beta function.

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For functions $f \in A$ of the form (1.1), we have

$$SL_\lambda f(z) = z + \sum_{m=2}^{\infty} B_\lambda(a,c,m,n) a_m z^m, \tag{1.4}$$

where $$B_\lambda(a,c,m,n) = \left[(1-\lambda)m^n + \lambda \frac{(a)_{m-1}}{(c)_{m-1}} \right]. \tag{1.5}$$

Here $(a)_m$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1, & for \quad m = 0 \\ a(a+1)(a+2)\cdots(a+m-1), & for \quad m \in N. \end{cases}$$

Now using the linear operator SL_λ we define the class $SL_\lambda^\alpha(a,c,n)$ consisting functions of the form (1.1) satisfying the condition

$$R\left\{ \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} - \alpha \right\} > \left| \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} - 1 \right|. \tag{1.6}$$

Silverman [9] defined the class $V(\theta_m)$ as the class of all functions in A such that $\arg a_m = \theta_m$ for all m. If further there exists a real number t such that $\theta_m + (m-1)t = \pi (\bmod 2\pi)$, then f is said to be in the class $V(\theta_m, t)$. The union of $V(\theta_m, t)$ taken overall possible sequences $\{\theta_m\}$ and all possible real numbers t is denoted by V.

Further, we define $VS_\lambda^\alpha(a,c,n,\beta) = S_\lambda^\alpha(a,c,n,\beta) \cap V$.

Definition1.1: A function $f \in V$ of the form (1.1) is in $VS_\lambda^\alpha(a,c,n,\beta)$ if f satisfies the analytic condition

$$R\left\{ \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} \right\} > \beta \left| \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} - 1 \right| + \alpha, \tag{1.7}$$

where $\alpha, \beta \geq 0$ and $z \in U$.

These classes stem essentially from the classes studied earlier by Vijaya and Murugusundaramoorthy [10].

2. MAIN RESULTS

Theorem 2.1: A function f of the form (1.1) is in $VS_\lambda^\alpha(a,c,n)$ if and only if

$$\sum_{m=2}^{\infty} (2m-1-\alpha) B_\lambda(a,c,m,n) |a_m| \leq 1-\alpha. \tag{2.1}$$

Proof: From (1.6), it suffices to show that

$$\left| \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} - 1 \right| \leq R\left\{ \frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)} - \alpha \right\}.$$

That is

$$\left|\frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)}-1\right|-R\left\{\frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)}-1\right\}$$
$$\leq 2\left|\frac{z(SL_\lambda f(z))'}{SL_\lambda f(z)}-1\right|$$
$$\leq 2\frac{\sum_{m=2}^{\infty}(m-1)B_\lambda(a,c,m,n)|a_m||z|^{m-1}}{1-\sum_{m=2}^{\infty}B_\lambda(a,c,m,n)|a_m||z|^{m-1}}.$$

Now the last expression is bounded by $(1-\alpha)$ if

$$\sum_{m=2}^{\infty}(2m-1-\alpha)B_\lambda(a,c,m,n)|a_m|\leq 1-\alpha.$$

Conversely, if $f\in VS_\lambda^\alpha(a,c,n)$ then by definition

$$\left|\frac{z+\sum_{m=2}^{\infty}mB_\lambda(a,c,m,n)a_m z^m}{z+\sum_{m=2}^{\infty}B_\lambda(a,c,m,n)a_m z^m}-1\right|\leq R\left\{\frac{z+\sum_{m=2}^{\infty}mB_\lambda(a,c,m,n)a_m z^m}{z+\sum_{m=2}^{\infty}B_\lambda(a,c,m,n)a_m z^m}-\alpha\right\}.$$

That is

$$\left|\frac{\sum_{m=2}^{\infty}(m-1)B_\lambda(a,c,m,n)a_m z^{m-1}}{1+\sum_{m=2}^{\infty}B_\lambda(a,c,m,n)a_m z^{m-1}}\right|\leq R\left\{\frac{(1-\alpha)+\sum_{m=2}^{\infty}(m-\alpha)B_\lambda(a,c,m,n)a_m z^{m-1}}{1+\sum_{m=2}^{\infty}B_\lambda(a,c,m,n)a_m z^{m-1}}\right\}$$

Since $f\in V$ and f lies in $V(\theta_m,t)$ for some sequence θ_m and a real number t such that $\theta_m+(m-1)t\equiv\pi(\bmod 2\pi)$ set $z=re^{it}$ in the above inequality

$$\left|\frac{\sum_{m=2}^{\infty}(m-1)B_\lambda(a,c,m,n)a_m r^{m-1}}{1+\sum_{m=2}^{\infty}B_\lambda(a,c,m,n)a_m r^{m-1}}\right|\leq R\left\{\frac{(1-\alpha)+\sum_{m=2}^{\infty}(m-\alpha)B_\lambda(a,c,m,n)a_m r^{m-1}}{1+\sum_{m=2}^{\infty}B_\lambda(a,c,m,n)a_m r^{m-1}}\right\}.$$

Letting $r\to 1$, leads the desired inequality

$$\sum_{m=2}^{\infty}(2m-1-\alpha)B_\lambda(a,c,m,n)|a_m|\leq 1-\alpha.$$

Corollary.2.2: If $f\in VS_\lambda^\alpha(a,c,n)$ then

$$|a_m|\leq\frac{1-\alpha}{(2m-1-\alpha)B_\lambda(a,c,m,n)},\qquad for\quad m\geq 2.$$

The sharpness follows for the function

$$f(z)=z+\sum_{m=2}^{\infty}\frac{1-\alpha}{(2m-1-\alpha)B_{\lambda}(a,c,m,n)},\qquad for\quad m\geq 2,\quad z\in U.$$

Similar to the proof of Theorem 2.1 we get the following result:

Theorem.2.3: A function f of the form (1.1) is in $VS_{\lambda}^{\alpha}(a,c,n,\beta)$ if and only if

$$\sum_{m=2}^{\infty}E_m B_{\lambda}(a,c,m,n)\left|a_m\right|\leq 1-\alpha, \tag{2.2}$$

where $E_m=m(\beta+1)-(\alpha+\beta)$.

The result obtained in our next Theorem unifies the radii results concerning close-to-convexity, starlikeness etc.

Theorem.2.4: Let $f\in VS_{\lambda}^{\alpha}(a,c,n,\beta)$. Then $\left|\frac{f*\Phi}{f*\Psi}-1\right|<1-\delta$, in $|z|<r$ with $\Phi(z)=z+\sum_{m=2}^{\infty}\gamma_m z^m$, and $\Psi(z)=z+\sum_{m=2}^{\infty}\mu_m z^m$, are analytic in U with the conditions $\gamma_m,\mu_m\geq 0,\quad \gamma_m\geq\mu_m$, for $m\geq 2$ and $f*\Psi\neq 0$, where

$$r=\inf_m\left[\frac{E_m B_{\lambda}(a,c,m,n)(1-\delta)}{(1-\alpha)[(\lambda_m-\mu_m)+\mu_m(1-\delta)]}\right]^{\frac{1}{m-1}},\quad m\geq 2. \tag{2.3}$$

Proof: Consider,

$$\begin{aligned}\left|\frac{f*\Phi}{f*\Psi}-1\right|&=\left|\frac{z-\sum_{m=2}^{\infty}\gamma_m a_m z^m}{z-\sum_{m=2}^{\infty}\mu_m a_m z^m}-1\right|\\&\leq\left|\frac{z-\sum_{m=2}^{\infty}\gamma_m a_m z^m-z+\sum_{m=2}^{\infty}\mu_m a_m z^m}{z-\sum_{m=2}^{\infty}\mu_m a_m z^m}\right|\\&\leq\frac{\sum_{m=2}^{\infty}a_m[\gamma_m-\mu_m]\,|z|^{m-1}}{1-\sum_{m=2}^{\infty}\mu_m a_m\,|z|^{m-1}}<1-\delta.\end{aligned}$$

$$\sum_{m=2}^{\infty}a_m[(\gamma_m-\mu_m)+(1-\delta)\mu_m]\leq 1-\delta,\qquad (|z|<r,\ 0\leq\delta<1), \tag{2.4}$$

where r is given by (2.3). From Theorem 2.3, (2.4) will be true if,

$$\frac{[(\gamma_m-\mu_m)+(1-\delta)\mu_m]}{1-\delta}\,|z|^{m-1}\leq\frac{E_m B_{\lambda}(a,c,m,n)(1-\delta)}{(1-\alpha)[(\gamma_m-\mu_m)+(1-\delta)\mu_m]},$$

that is, if

$$|z| = \left[\frac{E_m B_\lambda(a,c,m,n)(1-\delta)}{(1-\alpha)[(\gamma_m-\mu_m)+(1-\delta)\mu_m]}\right]^{\frac{1}{m-1}}. \tag{2.5}$$

As corollaries to the above Theorem we get the following result:

By choosing $\Phi(z) = \dfrac{z}{(1-z)^2}$ and $\Psi(z) = z$, we have

Corollary 2.5: Let the function f defined by (1.1) belongs to $VS_\lambda^\alpha(a,c,n,\beta)$. Then f is close-to-convex of order $\delta(0 \le \delta < 1)$, hence univalent in the disc $|z| < r_1$, where

$$r_1 = \inf_m \left[\frac{E_m B_\lambda(a,c,m,n)(1-\delta)}{(1-\alpha)m}\right]^{\frac{1}{m-1}}, \qquad m \ge 2. \tag{2.6}$$

The result is sharp.

For $\Phi(z) = \dfrac{z}{(1-z)^2}$ and $\Psi(z) = \dfrac{z}{1-z}$, we have

Corollary 2.6: Let the function f defined by (1.1) belongs to $VS_\lambda^\alpha(a,c,n,\beta)$. Then f is starlike of order $\delta(0 \le \delta < 1)$, hence univalent in the disc $|z| < r_2$, where

$$r_2 = \inf_m \left[\frac{E_m B_\lambda(a,c,m,n)(1-\delta)}{(1-\alpha)(m-\delta)}\right]^{\frac{1}{m-1}}, \qquad m \ge 2. \tag{2.7}$$

The result is sharp.

If $\Phi(z) = \dfrac{z+z^2}{(1-z)^3}$ and $\Psi(z) = \dfrac{z}{(1-z)^2}$, we have

Corollary 2.7: Let the function f defined by (1.1) belongs to $VS_\lambda^\alpha(a,c,n,\beta)$. Then f is convex of order $\delta(0 \le \delta < 1)$, hence univalent in the disc $|z| < r_3$, where

$$r_{3\backslash} = \inf_m \left[\frac{E_m B_\lambda(a,c,m,n)(1-\delta)}{m(1-\alpha)(m-\delta)}\right]^{\frac{1}{m-1}}, \qquad m \ge 2. \tag{2.8}$$

The result is sharp.

Using the coefficient inequality proved above we can easily prove the following growth and distortion theorem.

Theorem 2.8: Let f of the form (1.1) to be in $VS_\lambda^\alpha(a,c,n,\beta)$.then

$$r - \frac{1-\alpha}{E_2 B_\lambda(a,c,2,n)} r^2 \le |f(z)| \le r + \frac{1-\alpha}{E_2 B_\lambda(a,c,2,n)} r^2$$

and

$$1 - \frac{2(1-\alpha)}{E_2 B_\lambda(a,c,2,n)} r \le |f'(z)| \le 1 + \frac{2(1-\alpha)}{E_2 B_\lambda(a,c,2,n)} r.$$

The result is sharp.

Proof: Let f of the form (1.1) belongs to $VS_\lambda^\alpha(a,c,n,\beta)$.

$$|f(z)| = \left| z + \sum_{m=2}^{\infty} a_m z^m \right| \le |z| + |z|^2 \sum_{m=2}^{\infty} |a_m|,$$

since $f \in VS_\lambda^\alpha(a,c,n,\beta)$ and by Theorem 2.3, we have

$$E_2 B_\lambda(a,c,2,n) \sum_{m=2}^{\infty} |a_m| \le \sum_{m=2}^{\infty} E_m B_\lambda(a,c,m,n) |a_m| \le 1-\alpha.$$

Thus $|f(z)| \le |z| + \dfrac{1-\alpha}{E_2 B_\lambda(a,c,2,n)} |z|^2$.

That is

$$|f(z)| \le r + \frac{1-\alpha}{E_2 B_\lambda(a,c,2,n)} r^2,$$

similarly, we get

$$|f(z)| \le r - \frac{1-\alpha}{E_2 B_\lambda(a,c,2,n)} r^2.$$

On the other hand $f'(z) = 1 + \sum_{m=2}^{\infty} m a_m z^{m-1}$, and

$$|f'(z)| = 1 + \sum_{m=2}^{\infty} m |a_m| |z^{m-1}| \le 1 + |z| \sum_{m=2}^{\infty} m |a_m|,$$

since $f \in VS_\lambda^\alpha(a,c,n,\beta)$.

Then by Theorem 2.3 w have $\sum_{m=2}^{\infty} m |a_m| \le \dfrac{2(1-\alpha)}{E_2 B_\lambda(a,c,2,n)}$. Thus

$$|f'(z)| \le 1 + \frac{2(1-\alpha)}{E_2 B_\lambda(a,c,2,n)} r.$$ Similarly we get

$$|f'(z)| \ge 1 - \frac{2(1-\alpha)}{E_2 B_\lambda(a,c,2,n)} r.$$

This completes the proof.

Theorem 2.9: A function f of the form (1.1) belongs to $VS_\lambda^\alpha(a,c,n,\beta)$, with $\arg a_m = \theta_m$ where $[\theta_m + (m-1)t] = \pi (\mathrm{mod}\, 2\pi)$. Define and $f_1(z) = z$ and

$$f_m(z) = z + \frac{1-\alpha}{E_m B_\lambda(a,c,m,n)} e^{i\theta_m} z^m, \qquad m \ge 2, \quad z \in U.$$

Then $f \in VS_\lambda^\alpha(a,c,n,\beta)$ if and only if f expressed in the form $f(z) = \sum_{m=2}^{\infty} \mu_m f_m(z)$ where $\mu_m \ge 0$ and $\sum_{m=2}^{\infty} \mu_m = 1$.

Proof: If $f(z)=\sum_{m=2}^{\infty}\mu_m f_m(z)$ with $\sum_{m=2}^{\infty}\mu_m=1$ and $\mu_m\geq 0$ then

$$\sum_{m=2}^{\infty}E_mB_\lambda(a,c,m,n)\frac{1-\alpha}{E_mB_\lambda(a,c,m,n)}\mu_m$$

$$=\sum_{m=2}^{\infty}\mu_m(1-\alpha)=(1-\mu_1)(1-\alpha)\geq 1-\alpha.$$

Hence $f\in VS_\lambda^\alpha(a,c,n,\beta)$.

Conversely, let the function f defined by (1.1) be in the class $VS_\lambda^\alpha(a,c,n,\beta)$, since

$$|a_m|\leq\frac{1-\alpha}{E_mB_\lambda(a,c,m,n)},\qquad m=2,3,\cdots.$$

We may set $\mu_m=\dfrac{E_mB_\lambda(a,c,m,n)|a_m|}{1-\alpha},\quad m\geq 2$ and $\mu_1=1-\sum_{m=2}^{\infty}\mu_m$.

Then $f(z)=\sum_{m=1}^{\infty}\mu_m f_m(z)$, this completes the proof.

Lemma 2.10: **[3]** If for the functions f and g are analytic in U with $g\prec f$, then for $k>0$ and $0<r<1$

$$\int_0^{2\pi}|g(re^{i\theta})|^k d\theta\leq\int_0^{2\pi}|f(re^{i\theta})|^k d\theta.$$

In [6] Silverman found that the function $f_2(z)=z-\dfrac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve the integral means inequality, conjectured in [7] and [8], such that

$$\int_0^{2\pi}|f(re^{i\theta})|^k d\theta\leq\int_0^{2\pi}|f_2(re^{i\theta})|^k d\theta,\ \text{for all } f\in V,\ k>0\ and\ 0<r<1.$$

In [8] he also proved his conjecture for the subclasses $T^*(\beta)$ and $C(\beta)\ of\ T$.

Theorem 2.11: Let f of the form (1.1) belongs to $VS_\lambda^\alpha(a,c,n,\beta)$ and f_2 is defined by

$f_2(z)=z-\dfrac{1-\alpha}{E_2B_\lambda(a,c,2,n)}z^2$ then for $z=re^{i\theta}$, $0<r<1$, we have

$$\int_0^{2\pi}|f(z)|^k d\theta\leq\int_0^{2\pi}|f_2(z)|^k d\theta.$$

Proof: For $f(z)=z-\sum_{m=2}^{\infty}|a_m|z^m$, (2.9) is equivalent to prove that

$$\int_0^{2\pi}\left|1-\sum_{m=2}^{\infty}|a_m|z^{m-1}\right|^k d\theta\leq\int_0^{2\pi}\left|1-\frac{1-\alpha}{E_2B_\lambda(a,c,2,n)}z\right|^k d\theta.$$

By Lemma 2.10 it suffices to show that

$$1-\sum_{m=2}^{\infty}|a_m|z^{m-1} \prec 1-\frac{1-\alpha}{E_2B_\lambda(a,c,2,n)}z \text{ setting}$$

$$1-\sum_{m=2}^{\infty}|a_m|z^{m-1} = 1-\frac{1-\alpha}{E_2B_\lambda(a,c,2,n)}\omega(z)$$

And using (2.2) we obtain

$$\omega(z)=\left|\sum_{m=2}^{\infty}\frac{E_mB_\lambda(a,c,m,n)}{1-\alpha}|a_m|z^{m-1}\right|$$
$$\le |z|\sum_{m=2}^{\infty}\frac{E_mB_\lambda(a,c,m,n)}{1-\alpha}|a_m| \quad \le \quad |z|.$$

This completes the proof.

In Theorem 2.4, 2.8, 2.9 and 2.11 if we substitute $\beta=1$ we get the result for the class $VS_\lambda^\alpha(a,c,n)$.

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