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A UNIQUENESS RESULT RELATED TO CERTAIN NON-LINEAR DIFFERENTIAL POLYNOMIALS

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ABSTRACT

In this paper, we deal with some uniqueness question of meromorphic functions whose certain non-linear differential polynomials have a nonzero finite value, and obtain some results, which improve and generalize the related results due to I. Lahiri and R. Pal[4], X. M. Li and H. X. Yi[6] and A. Banerjee and P. Bhattacharjee[1].

1. INTRODUCTION

In this paper, by meromorphic function we will always mean meromorphic function in complex plane. We adopt the standard notations of Nevanlinna theory of meromorphic function as explained in [2], [7] and [8]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by T(r, h) the Nevanlinna characteristic of h and by S(r, h) any quantity satisfying S(r, h)=o{T(r, h)}, as $r \to \infty$ and $r \in E$.

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a -points with the same multiplicities. We say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities (see [8]). We say that a is a small function of f, if a is a meromorphic function satisfying T(r, a) = S(r, f) (see [8]). Let l be a positive integer or ∞. Next we denote by $E_{l)}(a;f)$ the set of those a-points of f in the coplex plane, where each point is of multiplicity $\leq l$ and counted according to its multiplicity. By $\bar{E}_{l)}(a;f)$ we denote the reduced form of $E_{l)}(a;f)$. If $\bar{E}_{l)}(a;f) = \bar{E}_{l)}(a;g)$, we say that a is a l-order pseudo common value of f and g (see[3]).

Obviously, if $E_{\infty)}(a;f) = E_{\infty)}(a;g)$ ($\bar{E}_{\infty)}(a;f) = \bar{E}_{\infty)}(a;g)$), resp. then f and g share a CM (IM, resp.).

In 2006, I. Lahiri and R. Pal [4] proved the following theorem.

Theorem A: Let f and g be two non-constant meromorphic functions, and let $n(\geq 14)$ be positive integer.

If $E_{3)}(1; f^n(f^3-1)f') = E_{3)}(1; g^n(g^3-1)g')$, then $f \equiv g$.

Theorem B: Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers satisfying n>3k+11 and max $\{\chi_1, \chi_2\} < 0$, where

$$\chi_1 = \frac{2}{n-2k+1} + \frac{2}{n+2k+1} + \frac{2k+1}{n+k+1} + 1 - \theta_{k)}(1,f) - \theta_{k-1)}(1,f)$$

and

$$\chi_2 = \frac{2}{n-2k+1} + \frac{2}{n+2k+1} + \frac{2k+1}{n+k+1} + 1 - \theta_{k)}(1,g) - \theta_{k-1)}(1,g).$$

If $\theta > 2/n$ and if $\{f^n(f-1)\}^{(k)} - P$ and $\{g^n(g-1)\}^{(k)} - P$ share 0CM, where P is nonzero polynomial, then $f \equiv g$.

Theorem C: Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers satisfying n>9k+20 and where max$\{\chi_1, \chi_2\}$<0, where χ_1, χ_2 are defined as in Theorem B.

If $\theta > 2/n$ and if $\{f^n(f-1)\}^{(k)} - P$ and $\{g^n(g-1)\}^{(k)} - P$ share 0 IM, where P is a nonzero polynomial, then $f \equiv g$.

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In 2011, A. Banerjee and P. Bhattacharjee [1] proved the following theorem.

Theorem D: Let f and g be two transcendental meromorphic functions, and let n, k ($\geq$ 1) and m ($\geq$ 2) be three positive integers. Suppose for two nonzero constants a and b, $E_{l)}\left(1;[f^n(af^m+b)]^{(k)}\right)=E_{l)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$. Then f $\equiv$ g or f $\equiv$ - g or $[f^n(af^m+b)]^{(k)}\ [g^n(ag^m+b)]^{(k)}\equiv 1$ provided one of the following holds:

(i) when $l\geq 3$ and n > 3k+m+8;
(ii) when $l=2$ and n > 4k+ $\frac{3m}{2}$+9;
(iii) when $l=1$ and n > 7k+3m+12.

When k=1 the possibility $[f^n(af^m+b)]^{(k)}\ [g^n(ag^m+b)]^{(k)}\equiv 1$ does not occur. Also the possibility f $\equiv$ -g arises only if n and m are both even.

Question: What can be said about the relationship between two meromorphic functions f and g, if the condition $E_{l)}\left(1;[f^n(af^m+b)]^{(k)}\right)=E_{l)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$ in Theorem B is replaced with the condition $\bar{E}_{l)}\left(1;[f^n(af^m+b)]^{(k)}\right)=\bar{E}_{l)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$.

We prove the following two theorems, which generalize and improves Theorem A, B, C and D and deals with above Question.

Theorem 1.1: Let f and g be two transcendental meromorphic functions, and let n, k($\geq$ 1) and m($\geq$ 2) be three positive integers with n > $\frac{13k+13m+28}{3}$ and a and b be nonzero constants.

If $\bar{E}_{l)}\left(1;[f^n(af^m+b)]^{(k)}\right)=\bar{E}_{l)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$ *and* $E_{1)}\left(1;[f^n(af^m+b)]^{(k)}\right)=E_{1)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$, *where* $l\geq 3$ *is an integer, then either* $f\equiv g$ *or* $f\equiv -g$ *or* $[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)}\equiv 1$.

The possibility $[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)}\equiv 1$ does not arise for k=1 and the possibility f $\equiv -$ g does not arise if n and m are both odd or if n is even and m is odd or if n is odd and m is even.

Theorem 1.2: Let f and g be two transcendental meromorphic functions, and let n, k($\geq$1) and m($\geq$2) be three positive integers with n> $\frac{3k+m+8}{3}$ and a and b be nonzero constants. If $\bar{E}_{l)}\left(1;[f^n(af^m+b)]^{(k)}\right)=\bar{E}_{l)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$ *and* $E_{2)}\left(1;[f^n(af^m+b)]^{(k)}\right)=E_{2)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$, where l$\geq$ 4 is an integer, then the conclusions of Theorem 1.1 still holds.

Remark 1: Theorem 1.2 is an improvement of Theorem A and Theorem D.

Remark 2: Theorem 1.2 is an improvement of Theorem C for m = 1, a = 1 and b = -1.

2. LEMMAS

In this section, we present some lemmas which are needed in the sequel.

Lemma 2.1: ([7]) Let f be a nonconstant meromorphic function and

$P(f)=a_0+a_1f+\cdots+a_nf^n$, where $a_0,a_1,\ldots,a_n$ are constants and $a_n\neq 0$. Then

$$T\left(r,P(f)\right)=nT(r,f)+S(r,f).$$

Lemma 2.2: ([5]) Let $\bar{E}_{l)}\left(1;[F*]^{(k)}\right)=\bar{E}_{l)}\left(1;[G*]^{(k)}\right)$, $E_{1)}\left(1;[F*]^{(k)}\right)=E_{1)}\left(1;[G*]^{(k)}\right)$ and H*$\neq$ 0, where l $\geq$ 3.

Then

$$\begin{aligned}T(r,F^*)\leq&\left(\frac{8}{3}+\frac{2}{3}k\right)\bar{N}(r,\infty;F*)+\frac{5}{3}\ \bar{N}(r,0;F*)+\frac{2}{3}\ N_k(r,o;F*)+N_{k+1}(r,o;F*)\\&+(k+2)\bar{N}(r,\infty;F*)+\bar{N}(r,0;G*)+N_{k+1}(r,o;G*)+S(r,F*)+S(r,G*)\end{aligned}$$

Where

$$H^*\equiv\left[\frac{(F*)^{(k+2)}}{(F*)^{(k+1)}}-\frac{2(F*)^{(k+1)}}{(F*)^{(k)}-1}\right]-\left[\frac{(G*)^{(k+2)}}{(G*)^{(k+1)}}-\frac{2(G*)^{(k+1)}}{(G*)^{(k)}-1}\right].$$

Lemma 2.3: ([5]) Let $\bar{E}_{l)}\left(1;[F*]^{(k)}\right)=\bar{E}_{l)}\left(1;[G*]^{(k)}\right)$ *and* $E_{1)}\left(1;[F*]^{(k)}\right)=E_{1)}\left(1;[G*]^{(k)}\right)$, where l ≥ 3.

If $\Delta_{1l}=\left(\frac{8}{3}+\frac{2}{3}k\right)\Theta(\infty,F*)+(k+2)\Theta(\infty,G*)+\frac{5}{3}\Theta(0,F*)+\Theta(0,G*)+\delta_{k+1}(0,F*)+\delta_{k+1}(0,G*)+\frac{2}{3}\delta_k(0,F*)$

$\Delta_{1l}>\frac{5}{3}k+9$, then either $[F*]^{(k)}[G*]^{(k)}\equiv 1$ or F* = G*.

Lemma 2.4: ([5]) Let $\bar{E}_{l)}\left(1;[F*]^{(k)}\right)=\bar{E}_{l)}\left(1;[G*]^{(k)}\right)$, $E_{2)}\left(1;[F*]^{(k)}\right)=E_{2)}\left(1;[G*]^{(k)}\right)$ and H*≠ 0, where l ≥ 4.

Then

$$T(r,F^*)+T(r,G^*)\le(k+4)\bar{N}(r,\infty;F*)+2\,\bar{N}(r,0;F*)+2N_{k+1}(r,o;F*)$$
$$+(k+4)\bar{N}(r,\infty;G*)+2\bar{N}(r,0;G*)+2N_{k+1}(r,o;G*)+S(r,F*)+S(r,G*)$$

Where H* is defined as Lemma 2.2 .

Lemma 2.5: ([5]) Let $\bar{E}_{l)}\left(1;[F*]^{(k)}\right)=\bar{E}_{l)}\left(1;[G*]^{(k)}\right)$ *and* $E_{2)}\left(1;[F*]^{(k)}\right)=E_{2)}\left(1;[G*]^{(k)}\right)$, where l ≥ 4.

If $\Delta_{2l}=\left(2+\frac{1}{2}k\right)\Theta(\infty,F*)+\left(\frac{1}{2}k+2\right)\Theta(\infty,G*)+\Theta(0,F*)+\Theta(0,G*)+\delta_{k+1}(0,F*)+\delta_{k+1}(0,G*)$

$\Delta_{2l}>k+5$, then either $[F*]^{(k)}[G*]^{(k)}\equiv 1$ or F* = G*.

Lemma 2.6: ([1]) Let f and g be two nonconstant meromorphic functions and a and b be nonzero constants. Then $[f^n(af^m+b)]^1[g^n(ag^m+b)]^1\neq 1$, where n, m ≥ 2 be two positive integers and n (≥ m+3).

3. PROOF OF THE THEOREM

Proof of Theorem 1.1: Let F* = $f^n(af^m+b)$, G* = $g^n(ag^m+b)$.

By Lemma 2.1, we get

(3.1) $\Theta(0,F*)=1-\lim_{r\to\infty}sup\frac{\bar{N}(r,0;F*)}{T(r,F*)}\ \ge\ \frac{n-1}{n+m}$

Similarly

(3.2) $\Theta(0,G*)\ge\frac{n-1}{n+m}$

(3.3) $\Theta(\infty,F*)=1-\lim_{r\to\infty}sup\frac{\bar{N}(r,\infty;F*)}{T(r,F*)}\ \ge\ \frac{n+m-1}{n+m}$

Similarly

(3.4) $\Theta(\infty,G*)\ge\frac{n+m-1}{n+m}$

(3.5) $\delta_{k+1}(0,F*)=1-\lim_{r\to\infty}sup\frac{N_{k+1}(r,o;F*)}{T(r,F*)}\ \ge\ \frac{n-k-1}{n+m}$

Similarly

(3.6) $\delta_{k+1}(0,G*)\ge\frac{n-k-1}{n+m}$, $\delta_k(0,F*)\ge\frac{n-k}{n+m}$ and $\delta_k(0,G*)\ge\frac{n-k}{n+m}$

From the condition of Theorem 1.1, we have

$\bar{E}_{l)}\left(1;[f^n(af^m+b)]^{(k)}\right)=\bar{E}_{l)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$ *and* $E_{1)}\left(1;[f^n(af^m+b)]^{(k)}\right)=E_{1)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$, *where* $l\ge 3$.

From (3.1) - (3.6) and Lemma 2.3, we have

$$\Delta_{1l}=\left(\frac{14}{3}+\frac{5}{3}k\right)\frac{n+m-1}{n+m}+\frac{8}{3}\frac{n-1}{n+m}+2\,\frac{n-k-1}{n+m}+\frac{2}{3}\frac{n-k}{n+m}$$

It is easily verified that if n>$\frac{13k+13m+28}{3}$, then $\Delta_{1l} > \frac{5}{3}k+9$. So by Lemma 2.3, we have $[F*]^{(k)}[G*]^{(k)} \equiv 1$ or F* ≡ G*. Also by Lemma 2.6 the case $[F*]^{(k)}[G*]^{(k)} \equiv 1$ does not arise for k = 1 and m ≥ 2.

Let F* ≡ G*, i.e.,

$$f^n(af^m+b) \equiv g^n(ag^m+b)$$

Clearly if n and m are both odd or if n is even and m is odd or if n is odd and m is even, then f ≡ $-$g contradicts F* ≡ G*. Let neither f ≡ g nor f ≡ $-$g. We put h = $\frac{g}{f}$. Then h ≠ 1 and h ≠ -1. Also F* ≡ G* implies

$$f^m=-\frac{b}{a}\frac{h^n-1}{h^{n+m}-1}.$$

Since f is non-constant it follows that h is non-constant. Again since f^m has no simple pole $h-u_r$ has no simple zero, where $u_r=exp\left(\frac{2\pi ir}{n+m}\right)$and r = 1, 2...n+m-1. Therefore either f ≡ g or f ≡ $-$g. This proves the theorem.

Proof of Theorem 1.2: From the condition of Theorem 1.2,

we have $\bar{E}_{l)}\left(1;[f^n(af^m+b)]^{(k)}\right)=\bar{E}_{l)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$
and $E_{2)}\left(1;[f^n(af^m+b)]^{(k)}\right)=E_{2)}\left(1;[g^n(ag^m+b)]^{(k)}\right)$, where l ≥ 4.

From (3.1)-(3.6) and Lemma 2.5, we have

$$\Delta_{2l}=(k+4)\frac{n+m-1}{n+m}+2\frac{n-1}{n+m}+2\,\frac{n-k-1}{n+m}.$$

It is easily verified that if n > $\frac{3k+m+8}{3}$, then Δ_{2l} > k+5. So by Lemma 2.5, we have $[F*]^{(k)}[G*]^{(k)} \equiv 1$ or F* ≡ G*.

Proceeding as in the proof of Theorem 1.1, we can get the conclusion of Theorem 1.2. Thus, we complete the proof of Theorem 1.2.

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