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EVALUATING THE EFFICIENCY OF DMUS WITH STOCHASTIC DATA USING BON FERRONI INEQUALITY VIA CCR MODEL

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ABSTRACT

Data envelopment analysis (DEA) is non-parametric method for evaluating the efficiency of the decision making units (DMUs) using mathematical programming. The models which exist are just for evaluating the performance of DMUs with deterministic data. Now the main point which can take managers attraction is that how we could evaluate the efficiency of DMUs if data were stochastically? In this article we propose a method for evaluating the efficiency of DMUs with stochastic inputs and outputs via CCR model using chance constrained programming and changing it to deterministic equivalent model. Finally, a numerical example is presented to show the application of this method.

Key Words: *DEA, Stochastic programming.*

1. INTRODUCTION

Stochastic programming approach is a method for modeling problems in which some of or all of the parameters of a problem expressed by random variables. These cases can appear frequently in our life such as determining the amount of profits from product sales, access to resources...

The main goal in stochastic programming is focusing on random variables that can be built with a model answer. The main idea in all the models of stochastic programming, such as Williams's method [1] is to change it into equivalent deterministic models. There are various methods for evaluating the efficiency of DMUs when the data are crisp such as Charnes, Cooper and Rodes [2] or Banker, Charnes, Cooper [3] method and also there are methods for evaluating the efficiency of DMUs with stochastic data Charnes, Cooper and Tintner [4-6] or Beale [5] which solve the quadratic form of stochastic programming or Dantzing method [7] for solving linear programming problems with uncertain data. Among the numerous models we considered the chance constrained programming. Recently formulation of the original models with the introduction of random inputs and outputs has been considered by different researchers. Cooper and Huang [8], Asgharian and Khodabakhshi [9] are working in this issue. There are also different methods for ranking the efficient DMUs with stochastic data, including Hosseinzadeh Lotfi and Nematollahi [10] ranking using coefficient variation also ranking using AP technique of Razavyan, and Tohidi [11].In this article we assume that all of the inputs and outputs of DMUs have random data. The main propose is to evaluate the efficiency of DMUs with stochastic data by using Bon Ferroni inequality via CCR model and changing the stochastic model into equivalent crisp model by using the chance constrained programming.

2. PRELIMINARS

Let us assume that there exist *n*, DMUs with *m* inputs and *s* outputs. Consider that $X_j = (x_{1j}, x_{2j},..., x_{mj})^T$ and $Y_j = (y_{1j}, y_{2j},..., y_{sj})^T$ are respectively input and output vectors of DMU_j , the production possibly set of CCR model defined as follows:

$$T_{CCR} = \left\{ (X, Y) \mid X \geq \sum_{j=1}^{n} \lambda_j X_j ,\ Y \leq \sum_{j=1}^{n} \lambda_j Y_j ,\ \lambda_j \geq 0 \right\}$$

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Defnition1 (Bon ferroni inequality): If the set of arbitrary events $A_1, A_2, \ldots A_n$ constitutes a partition of the sample space S and $A_1^{\,c}, A_2^{\,c}, \ldots, A_n^{\,c}$ are complements of events the following rule applied:

$$P(\cap_{i=1}^{n} A_i^c) \geq 1 - \sum_{i=1}^{n} P(A_i) \tag{1}$$

3. STOCHASTIC MODEL

Suppose that there exist n decision making unit with stochastic inputs and outputs. Consider that $\tilde{x}_j = (\tilde{x}_{1j}, \tilde{x}_{2j}, \ldots, \tilde{x}_{mj})^T$, $\tilde{y}_j = (\tilde{y}_{1j}, \tilde{y}_{2j}, \ldots, \tilde{y}_{sj})^T$, $j = 1,2,\ldots,n$ are the stochastic input and output vectors of DMU_j respectively and $\bar{x}_j = (\bar{x}_{1j}, \bar{x}_{2j}, \ldots, \bar{x}_{mj})^T$, $\bar{y}_j = (\bar{y}_{1j}, \bar{y}_{2j}, \ldots, \bar{y}_{sj})^T$, $j = 1,2,\ldots,n$ are deterministic equivalent input and output vectors respectively. It is assumed that all of inputs and outputs are normally distributed.

The CCR model for evaluating the efficiency of DMUs would be as follows:

$$
\begin{aligned}
&Min \quad \theta \\
&s.t \sum_{j=1}^{n} \lambda_j \, x_{ij} \leq \theta x_{io} \;, i = 1, \ldots, m \\
&\sum_{j=1}^{n} \lambda_j \, y_{rj} \geq y_{ro} \;, r = 1, \ldots, s \\
&\qquad \lambda_j \geq 0 \;, j = 1, \ldots, n
\end{aligned}
$$

Here we want to evaluate the efficiency of DMUs with stochastic data by using CCR model and the method of transforming the stochastic model into deterministic model will be explained.

$$
\begin{aligned}
&Min \quad \theta \\
&s.t \quad P\left(\sum_{j=1}^{n} \lambda_j \, \tilde{x}_{ij} \leq \theta \tilde{x}_{io} \,, \sum_{j=1}^{n} \lambda_j \, \tilde{y}_{rj} \geq \tilde{y}_{ro}\right) \leq 1 - \alpha \\
&\qquad \lambda_j \geq 0 \;, j = 1, \ldots, n
\end{aligned} \tag{2}
$$

Where in the above models, P means "probability" and $\alpha \in [0,1]$ is a level of error which is a predefined number, $1 - \alpha$Represents acceptance of constraints. We are explaining the steps of transforming to deterministic model now.

Suppose that:

$$
\begin{aligned}
&A_i = \sum_{j=1}^{n} \lambda_j \, \tilde{x}_{ij} \geq \theta \tilde{x}_{io} \qquad i = 1,2, \ldots, m \\
&A_{m+r} = \sum_{j=1}^{n} \lambda_j \, \tilde{y}_{rj} \leq \tilde{y}_{ro} \qquad r = 1, \ldots, s
\end{aligned} \tag{3}
$$

Through equations (1) and (3), we have**:**

$$P(\cap_{i=1}^{n} A_i^c) = P\left(\sum_{j=1}^{n} \lambda_j \, \tilde{x}_{ij} \leq \theta \tilde{x}_{io} \,, \sum_{j=1}^{n} \lambda_j \, \tilde{y}_{rj} \leq \tilde{y}_{ro}\right) \geq 1 - \alpha \tag{4}$$

And through equations (1) and (4), we can write:

$$P(A_i) \leq \frac{\alpha}{m+s} \tag{5}$$

$$Since \sum_{i=1}^{m+s} P(A_i) \leq \sum_{i=1}^{m+s} \frac{\alpha}{m+s} = \alpha$$

So we have:

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) \geq 1 - \sum_{i=1}^{m+s} p(A_i) \geq 1 - \alpha$$

From (5)

$$P(A_i) \leq \frac{\alpha}{m+s} \Rightarrow \begin{cases} P\left(\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \geq \theta \tilde{x}_{io}\right) \leq \dfrac{\alpha}{m+s} & (6) \\ P\left(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \leq \tilde{y}_{ro}\right) \leq \dfrac{\alpha}{m+s} & (7) \end{cases}$$

From (6):

$$P\left(\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \geq \theta \tilde{x}_{io}\right) \leq \frac{\alpha}{m+s} \Rightarrow P\left(\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} - \theta \tilde{x}_{io} \geq 0\right) = P\left(\underbrace{\sum_{j \neq o} \lambda_j \tilde{x}_{ij} + (\lambda_o - \theta)\tilde{x}_{io}}_{h_i} \geq 0\right) \leq \frac{\alpha}{m+s} ,$$

$i = 1, \dots, m$

Assuming:

$$h_i = \sum_{j \neq o} \lambda_j \tilde{x}_{ij} + (\lambda_o - \theta)\tilde{x}_{io} \quad , i = 1, \dots, m$$

We can obtain mean and variance of stochastic variableh_i through**:**

$$E(h_i) = E\left(\sum_{j \neq o} \lambda_j \tilde{x}_{ij} + (\lambda_o - \theta)\tilde{x}_{io}\right) = \sum_{j \neq o} \lambda_j \bar{x}_{ij} + (\lambda_o - \theta)\bar{x}_{io} \quad , i = 1, \dots, m$$

$$var(h_i) = \sum_{j \neq o} \sum_{k \neq o} \lambda_j \lambda_k cov(\tilde{x}_{ij}, \tilde{x}_{ik}) + 2(\lambda_o - \theta) \sum_{j \neq o} \lambda_j \, cov(\tilde{x}_{ij}, \tilde{y}_{io}) + (\lambda_o - \theta)^2 \, var(\tilde{x}_{io})$$

Consider $\sigma_i^I = \sqrt{var(h_i)}\, , i = 1, \dots, m$be Standard deviation of h_i so we have:

$$P\left(\frac{h_i - E(h_i)}{\sigma_i^I} \geq \frac{0 - E(h_i)}{\sigma_i^I}\right) \leq \frac{\alpha}{m+s}$$

$\frac{h_i - E(h_i)}{\sigma_i^I}$Have standard normal distribution like this:$\frac{h_i - E(h_i)}{\sigma_i^I} \sim N(0,1)$

However, when Z has standard normal distribution we can write $P(Z \geq -z) = P(Z \leq z) = \phi(z)$ and:

$$P\left(\frac{h_i - E(h_i)}{\sigma_i^I} \geq \frac{-E(h_i)}{\sigma_i^I}\right) \leq \frac{\alpha}{m+s} \Rightarrow P\left(\frac{h_i - E(h_i)}{\sigma_i^I} \leq \frac{E(h_i)}{\sigma_i^I}\right) \leq \frac{\alpha}{m+s}$$
$$\Rightarrow \phi\left(\frac{E(h_i)}{\sigma_i^I}\right) \leq \frac{\alpha}{m+s}$$

Suppose$\phi\left(-K_{\frac{\alpha}{m+s}}\right) = \frac{\alpha}{m+s}$, since ϕ is an inverse able function we have:

$$\phi\left(\frac{E(h_i)}{\sigma_i^I}\right) \leq \frac{\alpha}{m+s} = \phi(-K_{\frac{\alpha}{m+s}}) \Rightarrow \frac{E(h_i)}{\sigma_i^I} \leq -K_{\frac{\alpha}{m+s}} \Rightarrow E(h_i) \leq -K_{\frac{\alpha}{m+s}} \sigma_i^I$$

$$\sum_{j \neq o} \lambda_j \bar{x}_{ij} + (\lambda_o - \theta)\bar{x}_{io} \leq -K_{\frac{\alpha}{m+s}} \sigma_i^I \Rightarrow \sum_{j=1}^{n} \lambda_j \bar{x}_{ij} + K_{\frac{\alpha}{m+s}} \sigma_i^I \leq \theta \bar{x}_{io} \quad (8)$$

If the same explained procedure applied for outputs too, we have:

$$P\left(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \leq \tilde{y}_{ro}\right) \leq \frac{\alpha}{m+s} \Rightarrow P\left(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \tilde{y}_{ro} \leq 0\right) \leq \frac{\alpha}{m+s}$$

$$\Rightarrow P\left(\underbrace{\sum_{j\neq o}\lambda_j\,\tilde{y}_{rj}+(\lambda_o-1)\tilde{y}_{ro}}_{h_r}\leq 0\right)\leq\frac{\alpha}{m+s}\quad,r=1,\dots,s \tag{9}$$

Assuming: $h_r=\sum_{j\neq o}\lambda_j\,\tilde{y}_{rj}+(\lambda_o-1)\tilde{y}_{ro},r=1,\dots,s$ We can obtain mean and variance of stochastic variable h_r through:

$$E(h_r)=E\left(\sum_{j\neq o}\lambda_j\,\tilde{y}_{rj}+(\lambda_o-1)\tilde{y}_{ro}\right)=\sum_{j\neq o}\lambda_j\,\bar{y}_{rj}+(\lambda_o-1)\bar{y}_{ro}\quad,r=1,2,\dots,s \tag{10}$$

$$var(h_r)=var\left(\sum_{j\neq o}\lambda_j\,\tilde{y}_{rj}+(\lambda_o-1)\tilde{y}_{ro}\right)=\sum_{j\neq o}\sum_{k\neq o}\lambda_j\,\lambda_k\,cov(\tilde{y}_{rj},\tilde{y}_{rk})+2(\lambda_o-1)$$

$$\sum_{j\neq o}\lambda_j\,cov(\tilde{y}_{rj},\tilde{y}_{ro})+(\lambda_o-1)^2\,var(\tilde{y}_{ro}) \tag{11}$$

Consider $\sigma_r^o=\sqrt{var(h_r)}\,,r=1,\dots,s$ be Standard deviation of h_r so from (9) we have:

$$P\left(\frac{h_r-E(h_r)}{\sigma_r^o}\leq\frac{0-E(h_r)}{\sigma_r^o}\right)\leq\frac{\alpha}{m+s}\Rightarrow\phi\left(-\frac{E(h_r)}{\sigma_r^o}\right)\leq\frac{\alpha}{m+s} \tag{12}$$

Suppose $\phi\left(-K_{\frac{\alpha}{m+s}}\right)=\frac{\alpha}{m+s}$, and (12) then $\phi\left(-\frac{E(h_r)}{\sigma_r^o}\right)\leq\phi(-K_{\frac{\alpha}{m+s}})$, since ϕ is an inverse able function we have:

$$-\frac{E(h_r)}{\sigma_r^o}\leq -K_{\frac{\alpha}{m+s}}\;\Rightarrow E(h_r)\geq K_{\frac{\alpha}{m+s}}\sigma_r^o$$

From (10):

$$\sum_{j\neq o}\lambda_j\,\bar{y}_{rj}+(\lambda_o-1)\bar{y}_{ro}\geq K_{\frac{\alpha}{m+s}}\sigma_r^o\Rightarrow\sum_{j=1}^{n}\lambda_j\,\bar{y}_{rj}-K_{\frac{\alpha}{m+s}}\sigma_r^o\geq\bar{y}_{ro},r=1,\dots,s \tag{13}$$

From (2), (8), and (13) deterministic CCR model will be as follows:

$$\begin{aligned}&Min\quad\theta\\&s.t\sum_{j=1}^{n}\lambda_j\,\bar{x}_{ij}+K_{\frac{\alpha}{m+s}}\sigma_i^I\leq\theta\bar{x}_{io}\quad,i=1,\dots,m\\&\sum_{j=1}^{n}\lambda_j\,\bar{y}_{rj}-K_{\frac{\alpha}{m+s}}\sigma_r^o\geq\bar{y}_{ro}\qquad,r=1,\dots,s\\&\qquad\lambda_j\geq 0\,,j=1,\dots,n\end{aligned} \tag{14}$$

4. NUMERICAL EXAMPLE

In this section, we consider five banks with two stochastic inputs and outputs. Suppose $\tilde{X}_1$ be "payable profit", $\tilde{X}_2$ be "personal", $\tilde{Y}_1$ be "facilities" and $\tilde{Y}_2$ be "Received profit" of bank. All of the inputs and outputs are normally distributed with means and variances as shown in tables 1 to 4.

Table1. Mean of inputs (payable profit and personal)

$E\left(\tilde{x}_{ij}\right)=\bar{x}_{ij}$	A	B	C	D	E
$E\left(\tilde{x}_{1j}\right)=\bar{x}_{1j}$	6214.71	4937.71	16264.77	3187.22	10992.61
$E\left(\tilde{x}_{2j}\right)=\bar{x}_{2j}$	13.15	12.43	13.87	16.50	11.88

Table 2. Mean of outputs (Facilities and Received profit)

$E\left(\tilde{y}_{rj}\right)=\bar{y}_{rj}$	A	B	C	D	E
$E\left(\tilde{y}_{1j}\right)=\bar{y}_{1j}$	282125.65	180786.77	271150.02	8554755.15	862602.45
$E\left(\tilde{y}_{2j}\right)=\bar{y}_{2j}$	40742.8	9441.14	16774.20	84894.71	157820.95

Table 3. Variance of inputs (payable profit and personal)

$var(\tilde{x}_{ij})$	A	B	C	D	E
$var(\tilde{x}_{1j})$	61775357.94	37810008.77	28165380.94	18474675.91	180617623.7
$var(\tilde{x}_{2j})$	112.29	1.93	4.08	6.49	3.07

Table 4. Variance of outputs (Facilities and Received profit)

$var(\tilde{y}_{rj})$	A	B	C	D	E
$var(\tilde{y}_{1j})$	10438342998	1157943092	901499969.9	15248173088	46223196999
$var(\tilde{y}_{2j})$	319349741.2	102193533.9	336854494.1	9055718955	31520733945

The covariance of input and output values in tables (5) to (8) is listed.

Table 5. Covariance of the first input $\mathrm{cov}(X_1, X_1)$

$cov(\tilde{x}_{1j},\tilde{x}_{1j})$	$\tilde{x}_{11}$	$\tilde{x}_{12}$	$\tilde{x}_{13}$	$\tilde{x}_{14}$	$\tilde{x}_{15}$
$\tilde{x}_{11}$	59201384.69	46273169.87	39947161.84	32328417.12	104738576.3
$\tilde{x}_{12}$	46273169.87	36234591.73	31254093.77	25257222.19	82060967.48
$\tilde{x}_{13}$	39947161.84	31254093.77	26991823.4	21840390.15	70772451.89
$\tilde{x}_{14}$	32328417.12	25257222.19	21840390.15	17704897.74	57179560.38
$\tilde{x}_{15}$	104738576.3	82060967.48	70772451.89	57179560.38	185942039.4

Table 6. Covariance of the second input $\mathrm{cov}(X_2, X_2)$

$cov(\tilde{x}_{2j},\tilde{x}_{2j})$	$\tilde{x}_{21}$	$\tilde{x}_{22}$	$\tilde{x}_{23}$	$\tilde{x}_{24}$	$\tilde{x}_{25}$
$\tilde{x}_{21}$	11.78	1.20	1.52	1.27	1.57
$\tilde{x}_{22}$	1.20	1.77	2.56	2.58	2.21
$\tilde{x}_{23}$	1.52	2.56	3.91	4.37	3.28
$\tilde{x}_{24}$	1.27	2.58	4.37	6.22	3.36
$\tilde{x}_{25}$	1.57	2.21	3.28	3.36	2.95

Table 7. Covariance of the first output $\mathrm{cov}(Y_1, Y_1)$

$cov(\tilde{y}_{1j}, \tilde{y}_{1j})$	$\tilde{y}_{11}$	$\tilde{y}_{12}$	$\tilde{y}_{13}$	$\tilde{y}_{14}$	$\tilde{y}_{15}$
$\tilde{y}_{11}$	9984501998	-2480925023	-1065092995	-3200565967	-4415922686
$\tilde{y}_{12}$	-2480925023	1107597740	552393950.1	2072749896	2454473916
$\tilde{y}_{13}$	-1065092995	552393950.1	862304319	3142740309	4426989120
$\tilde{y}_{14}$	-3200565967	2072749896	2072749896	14585209041	21495611751
$\tilde{y}_{15}$	-4415922686	2454473916	4426989120	21495611751	44213492781

Table 8. Covariance of the second output $\mathrm{cov}(Y_2, Y_2)$

$cov(\tilde{y}_{2j}, \tilde{y}_{2j})$	$\tilde{y}_{21}$	$\tilde{y}_{22}$	$\tilde{y}_{23}$	$\tilde{y}_{24}$	$\tilde{y}_{25}$
$\tilde{y}_{21}$	305464969.9	165211140.2	1628811015	1571499966	2921229784
$\tilde{y}_{22}$	165211140.2	97750336.75	175234655.6	878611276.7	3063433415
$\tilde{y}_{23}$	1628811015	175234655.6	322208646.5	1646401317	3063433415
$\tilde{y}_{24}$	1571499966	878611276.7	1646401317	8661992044	16151354522
$\tilde{y}_{25}$	2921229784	3063433415	3063433415	16151354522	30150267252

Considering α=0.4, 0.9, using model (14) and Lingo the following results can occur:

DMU_j	A	B	C	D	E
$\alpha = 0.4$	1.000	0.9999	0.9998	1.000	1.000
$\alpha = 0.9$	1,000	0.9999	1.000	1.000	1.000

5. CONCLUSION

Previous models of data envelopment analysis which evaluate the relative efficiency of DMUs are restricted to crisp inputs and outputs. Finally the created model is linear programming. Here a method is proposed for evaluating the efficiency of DMUs by using Bon Ferroni inequality when all the data are stochastic. The created model is a nonlinear programming and for different values of α the relative efficiency of DMUs can be evaluated.

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