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Theorem on Fixed Points in n- Metric Spaces

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ABSTRACT

In this paper, we prove a fixed point theorem on n- metric spaces. Here we are dealing with n-metric spaces and n mapping. This paper result extends the result of L.Kikina [1] from three metric spaces to n-metric spaces and also it generalizes the results of S.Č.Nešić [2] from two metric spaces to n-metric spaces.

Keywords: *Complete metric space, fixed point.*

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1. INTRODUCTION

The following Fixed point theorem on two complete metric spaces proved by B. Fisher [3].

Theorem1.1: Let (X, d) and (Y, e) be complete metric spaces. If T is a mapping of X into Y and S is mapping of Y into X satisfying the Inequality

$$e(Tx,TSy) \le c\max\{d(x,Sy), e(y,Tx), e(y,TSy)\}$$
$$d(Sy,STx) \le c\max\{e(y,Tx), d(x,Sy), d(x,STx)\}$$

For all x in X and y in Y. where $0 \le c < 1$, then ST has a unique fixed point z in X and TS has unique fixed point w in y. Further, Tz = w and Sw = z.

In 1991, V. Popa [5] gave a result on two complete metric spaces.

Theorem1.2: Let (X, d) and (Y, ρ) be complete metric spaces, if T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities

$$d^2(Sy,STx) \le c_1\max\{\rho(y,Tx)\, d(x,Sy), \rho(y,Tx)d(x,STx),\ d(x,Sy)\, d(x,STx)\}$$
$$\rho^2(Tx,TSy) \le c_2\max\{d(x,Sy)\rho(y,Tx), d(x,Sy)\, , \rho(y,TSy), \rho(y,Tx)\rho(y,TSy)\}$$

For all x in X and y in Y where $0 \le c_1.c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further Tz = w and Sw = z.

The following Fixed point theorem on three complete metric spaces proved by R.K .Jain [4].

Theorem1.3: Let (X, d_1), (Y, d_2) and (Z, d_3) be complete metric spaces. If T is continuous mapping X into Y, S is continuous mapping Y into Z and R is mapping of Z into X satisfying the following inequalities

$$d_1(RSTx, RST\,x') \le c\max\{d_1(x,x'), d_1(x,RSTx), d_1(x',RSTx'), d_2(Tx,Tx'), d_3(STx,STx')\}$$
$$d_2(TRSy,TRSy') \le c\max\{d_2(y,y'), d_2(y,TRSy), d_2(y',TRS\,y'), d_3(Sy,Sy'), d_1(RSy,RSy')\}$$
$$d_3(STRz,STRz') \le c\max\{d_3(z,z'), d_3(z,STRz), d_3(z',STR\,z'), d_1(Rz,Rz'), d_2(TRz,TRz')\}$$

for all x,x' in X , y,y' in Y and z,z'in Z, where $0 \le c < 1$, then RST has unique fixed point u in X , TRS has unique fixed point v in Y and STR has fixed point w in Z . Further, Tu = v, Sv = w and Rw = u.

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2. PRELIMINARIES

Definition 2.1 A Metric space is an ordered pair (X, d) where X is a non empty Set and d is a metric on X or also called distance function or simple distance, i.e a function d:XxX →R such that

i. d(x,y) ≥ 0 (non -negative)

ii. d(x,y)= 0 iff x=y

iii. d(x,y)= d(y, x) (symmetric)

iv. d(x,y)≤ d(x, z)+ d(z, y) (triangle inequality)

Definition2.2: Let (X, d) be a Metric Space. A Sequence $\{x_n\}$ in X is said to converge to point x in X iff the following criterion is satisfied.

For each ε >0, there exit a positive integer $n_0(\varepsilon)$, Such that

$$d(x_n, x) < \varepsilon, \text{ for all } n \geq n_0$$

Definition2.3: A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy Sequence iff for each ε>0, there exist a positive integer number $n_o(\varepsilon)$ such that

$$d(x_m, x_n) < \varepsilon, \text{ for all } n, m \geq n_0$$

Definition2.4: A metric space (X, d) is said to be complete iff every Cauchy sequence in X converges to point of X.

Let (X_i, d_i) be complete metric spaces where $i = 1, 2, 3, \ldots, n$. If A_i is mapping of X_i to X_{i+1} where $i = 1, 2, 3, \ldots, n-1$ and A_n is mapping of x_n to x_1.

we denote

$$M_1(x^1, x^2) = \{d_1^p(x^1, A_nA_{n-1}\ldots A_2x^2), d_1^p(x^1, A_nA_{n-1}\ldots A_2A_1x^1), d_2^p(x^2, A_1x^1)\} \tag{2.1}$$

$$M_2(x^2, x^3) = \{d_2^p(x^2, A_1A_n\ldots A_3x^3), d_2^p(x^2, A_1A_n\ldots A_2x^2), d_3^p(x^3, A_2x^2)\} \tag{2.2}$$

$$M_3(x^3, x^4) = \{d_3^p(x^3, A_2A_1A_n\ldots A_4x^4), d_3^p(x^3, A_2A_1A_n\ldots A_3x^3), d_4^p(x^4, A_3x^3)\} \tag{2.3}$$

And so on,

$$M_n(x^n, x^1) = \{d_n^p(x^n, A_{n-1}\ldots A_2A_1x^1), d_n^p(x^n, A_{n-1}\ldots A_1A_nx^n), d_1^p(x^1, A_nx^n)\} \tag{2.4}$$

Let $F : [0, \infty] \to R^+$ be continuous mapping in 0 with $F(0) = 0$

In this paper we prove a fixed point theorem on n metric spaces and some corollaries which extend the result of L.Kikina [1] from three metric spaces to n-metric spaces and a version of Vishal Gupta et al [6-7].

3. MAIN RESULT

Theorem 3.1: Let (X_i, d_i) be complete metric spaces where $i = 1, 2, 3, \ldots, n$. If A_i is mapping of X_i to X_{i+1} where $i = 1, 2, 3, \ldots, n-1$ and A_n is mapping of x_n to x_1 satisfying the following inequalities.

$$d_1^p(A_nA_{n-1}\ldots A_2x^2, A_nA_{n-1}\ldots A_2A_1x^1) \leq c\max M_1(x^1, x^2) + F(\min M_1) \tag{3.1}$$

$$d_2^p(A_1A_n\ldots A_4A_3x^3, A_1A_n\ldots A_3A_2x^2) \leq c\max M_2(x^2, x^3) + F(\min M_2) \tag{3.2}$$

$$d_3^p(A_2A_1A_n\ldots A_5A_4x^4, A_2A_1A_n\ldots A_4A_3x^3) \leq c\max M_3(x^3, x^4) + F(\min M_3) \tag{3.3}$$

So continuously like above.

$$d_n^p(A_{n-1}A_{n-2}\ldots A_2A_1x^1, A_{n-1}A_{n-2}\ldots A_2A_1A_nx^n) \leq c\max M_n(x^n, x^1) + F(\min M_n) \tag{3.4}$$

$\forall\, x^1 \in X_1, x^2 \in X_2, \ldots, x^n \in X_n$, where $0 \le c < 1$. Then $A_n A_{n-1} \ldots A_2 A_1$ has a unique fixed point $\beta_1 \in X_1$, $A_1 A_n \ldots A_3 A_2$ has a unique fixed point $\beta_2 \in X_2$, $A_2 A_1 \ldots A_4 A_3$ has a unique fixed point $\beta_3 \in X_3$ and so on $A_{n-1} A_{n-2} \ldots A_1 A_n$ has a unique fixed point $\beta_n \in X_n$. Further, $A_1(\beta_1) = \beta_2, A_2(\beta_2) = \beta_3, A_3(\beta_3) = \beta_4, \ldots, A_{n-1}(\beta_{n-1}) = \beta_n, A_n(\beta_n) = \beta_1$

Proof: Let x_0^1 be an arbitrary point in X_1, let define sequence $\{x_m^1\}$, $\{x_m^2\}, \ldots, \{x_m^n\}$ in $X_1, X_2, \ldots, X_n$ respectively by

$$\begin{aligned}
\left(A_n A_{n-1} \ldots A_3 A_2 A_1\right)^m x_0^1 &= x_m^1, \\
x_m^2 &= A_1\left(x_{m-1}^1\right), \\
x_m^3 &= A_2\left(x_m^2\right) \\
&\vdots \\
x_m^n &= A_{n-1}\left(x_m^{n-1}\right) \\
x_m^1 &= A_n\left(x_m^n\right) \qquad \text{for } m = 1, 2, 3, \ldots
\end{aligned}$$

We will assume that $x_m^1 \ne x_{m+1}^1, x_m^2 \ne x_{m+1}^2$ and so on $x_m^n \ne x_{m+1}^n$ for all m. Otherwise, if $x_m^1 = x_{m+1}^1$ for some m, then $x_m^2 = x_{m+1}^2$, $x_m^3 = x_{m+1}^3$ and so on $x_m^n = x_{m+1}^n$, we could put $x_m^1 = \beta_1$, $x_{m+1}^2 = \beta_2$ and so on $x_{m+1}^n = \beta_n$.

First, we prove the sequences $\{x_m^1\}$, $\{x_m^2\}, \ldots, \{x_m^n\}$ are cauchy sequences.

Taking $x^1 = x_m^1$, $x^2 = x_m^2$ in (2.1) and (3.1) , we obtain

$$\begin{aligned}
M_1(x_m^1, x_m^2) &= \{d_1^p(x_m^1, A_n A_{n-1} \ldots A_2 x_m^2), d_1^p(x_m^1, A_n A_{n-1} \ldots A_2 A_1 x_m^1), d_2^p(x_m^2, A_1 x_m^1)\} \\
&= \{d_1^p(x_m^1, x_m^1), d_1^p(x_m^1, x_{m+1}^1), d_2^p(x_m^2, x_{m+1}^2)\} \\
&= \{0, d_1^p(x_m^1, x_{m+1}^1), d_2^p(x_m^2, x_{m+1}^2)\}
\end{aligned}$$

$$\begin{aligned}
d_1^p(x_m^1, x_{m+1}^1) &= d_1^p(A_n A_{n-1} \ldots A_2 x_m^2, A_n A_{n-1} \ldots A_2 A_1 x_m^1) \\
&\le c \max M_1(x_m^1, x_m^2) + F(\min M_1(x_m^1, x_m^2)) \\
&= c \max\{0, d_1^p(x_m^1, x_{m+1}^1), d_2^p(x_m^2, x_{m+1}^2)\} + F(0) \\
&= c d_2^p(x_m^2, x_{m+1}^2)
\end{aligned}$$

Since, if $\max M_1(x_m^1, x_m^2) = d_1^p(x_m^1, x_{m+1}^1)$, then

$$d_1^p(x_m^1, x_{m+1}^1) \le c d_1^p(x_m^1, x_{m+1}^1)$$

It follows $x_m^1 = x_{m+1}^1$ since $0 \le c < 1$, so

$$d_1^p(x_m^1, x_{m+1}^1) \le c d_2^p(x_m^2, x_{m+1}^2) \tag{3.5}$$

Taking $x^2 = x_m^2, x^3 = x_{m-1}^3$ in (2.2) and (3.2), we get

$$\begin{aligned}
M_2(x_m^2, x_{m-1}^3) &= \{d_2^p(x_m^2, A_1 A_n \ldots A_3 x_{m-1}^3), d_2^p(x_m^2, A_1 A_n \ldots A_2 x_m^2), d_3^p(x_{m-1}^3, A_2 x_m^2)\} \\
&= \{d_2^p(x_m^2, x_m^2), d_2^p(x_m^2, x_{m+1}^2), d_3^p(x_{m-1}^3, x_m^3)\} \\
&= \{0, d_2^p(x_m^2, x_{m+1}^2), d_3^p(x_{m-1}^3, x_m^3)\}
\end{aligned}$$

$$
\begin{aligned}
d_2^p(x_m^2, x_{m+1}^2) &= d_2^p(A_1A_n \dots A_4A_3x_{m-1}^3, A_1A_n \dots A_3A_2x_m^2) \\
&\le c \max M_2(x_m^2, x_{m-1}^3) + F(\min M_2(x_m^2, x_{m-1}^3)) \\
&= c \max\{0, d_2^p(x_m^2, x_{m+1}^2), d_3^p(x_{m-1}^3, x_m^3)\} + F(0)
\end{aligned}
$$

Since $0 \le c < 1$, we get

$$d_2^p(x_m^2, x_{m+1}^2) \le c d_3^p(x_{m-1}^3, x_m^3) \tag{3.6}$$

Taking $x^3 = x_m^3, x^4 = x_{m-1}^4$ in (2.3) and (3.3), we get

$$
\begin{aligned}
M_3(x_m^3, x_{m-1}^4) &= \{d_3^p(x_m^3, A_2A_1A_n \dots A_4x_{m-1}^4), d_3^p(x_m^3, A_2A_1A_n \dots A_3x_m^3), d_4^p(x_{m-1}^4, A_3x_m^3)\} \\
&= \{d_3^p(x_m^3, x_m^3), d_3^p(x_m^3, x_{m+1}^3), d_4^p(x_{m-1}^4, x_m^4)\} \\
&= \{0, d_3^p(x_m^3, x_{m+1}^3), d_4^p(x_{m-1}^4, x_m^4)\}
\end{aligned}
$$

$$
\begin{aligned}
d_3^p(x_m^3, x_{m+1}^3) &= d_3^p(A_2A_1A_n \dots A_5A_4x_{m-1}^4, A_2A_1A_n \dots A_4A_3x_m^3) \\
&\le c \max M_3(x_m^3, x_{m-1}^4) + F(\min M_3(x_m^3, x_{m-1}^4)) \\
&= c \max\{0, d_3^p(x_m^3, x_{m+1}^3), d_4^p(x_{m-1}^4, x_m^4)\} + F(0) \\
&= c d_4^p(x_{m-1}^4, x_m^4)
\end{aligned}
$$

Replacing m with m-1 we obtain

$$d_3^p(x_{m-1}^3, x_m^3) \le c d_4^p(x_{m-2}^4, x_{m-1}^4) \tag{3.7}$$

Continuously like above, taking $x^n = x_m^n, x^1 = x_{m-1}^1$ in (2.4) and (3.4) we obtain

$$
\begin{aligned}
M_n(x_m^n, x_{m-1}^1) &= \{d_n^p(x_m^n, A_{n-1} \dots A_2A_1x_{m-1}^1), d_n^p(x_m^n, A_{n-1} \dots A_1A_nx_m^n), d_1^p(x_{m-1}^1, A_nx_{m-1}^1)\} \\
&= \{d_n^p(x_m^n, x_m^n), d_n^p(x_m^n, x_{m+1}^n), d_1^p(x_{m-1}^1, x_m^1)\} \\
&= \{0, d_n^p(x_m^n, x_{m+1}^n), d_1^p(x_{m-1}^1, x_m^1)\}
\end{aligned}
$$

$$
\begin{aligned}
d_n^p(x_m^n, x_{m+1}^n) &= d_n^p(A_{n-1}A_{n-2} \dots A_2A_1x_{m-1}^1, A_{n-1}A_{n-2} \dots A_2A_1A_nx_m^n) \\
&\le c \max M_n(x_m^n, x_{m-1}^1) + F(\min M_n(x_m^n, x_{m-1}^1)) \\
&= c \max\{0, d_n^p(x_m^n, x_{m+1}^n), d_1^p(x_{m-1}^1, x_m^1)\} + F(0) \\
&= c d_1^p(x_{m-1}^1, x_m^1)
\end{aligned}
$$

Replacing m with m-n+2, we get

$$d_n^p(x_{m-n+2}^n, x_{m-n+3}^n) \le c d_1^p(x_{m-n+1}^1, x_{m-n+2}^1) \tag{3.8}$$

Using (3.5), (3.6), (3.7) and (3.8), we get

$$
\begin{aligned}
d_1^p(x_m^1, x_{m+1}^1) &\le c d_2^p(x_m^2, x_{m+1}^2) \le c^2 d_3^p(x_{m-1}^3, x_m^3) \le \dots \le c^{n-1} d_n^p(x_{m-n+2}^n, x_{m-n+3}^n) \le c^n d_1^p(x_{m-n+1}^n, x_{m-n+2}^n) \\
&\le c^{2n} d_1^p(x_{m-2n+2}^n, x_{m-2n+3}^n) \\
&\le \dots \le \begin{cases} c^{nk} d_1^p(x_1^1, x_2^1),, m = (n-1)k + 1 \\ c^{nk} d_1^p(x_0^1, x_1^1),, m = (n-1)k \end{cases}
\end{aligned}
$$

Since $0 \le c < 1$, the sequences $\{x_m^1\}$, $\{x_m^2\},\ldots,\{x_m^n\}$ are cauchy sequences. Since (X_i, d_i) be complete metric spaces where $i = 1,2,3,\ldots,n$.

$$\lim_{m\to\infty} x_m^1 = \beta_1 \in X_1$$
$$\lim_{m\to\infty} x_m^2 = \beta_2 \in X_2$$
$$\lim_{m\to\infty} x_m^3 = \beta_3 \in X_3$$
$$.$$
$$.$$
$$.$$
$$\lim_{m\to\infty} x_m^n = \beta_n \in X_n$$

Now taking $x^1 = x_m^1$ and $x^2 = \beta_2$ in the inequality (3.1) we obtain

$$\begin{aligned} d_1^p(A_nA_{n-1}\ldots A_2\beta_2, x_{m+1}^1) &= d_1^p(A_nA_{n-1}\ldots A_2\beta_2, A_nA_{n-1}\ldots A_2A_1x_m^1) \\ &\le c\max M_1(x_m^1,\beta_2) + F(\min M_1(x_m^1,\beta_2)) \end{aligned} \tag{3.9}$$

Where $M_1(x_m^1,\beta_2) = \{d_1^p(x_m^1, A_nA_{n-1}\ldots A_2\beta_2), d_1^p(x_m^1, A_nA_{n-1}\ldots A_2A_1x_m^1), d_2^p(\beta_2, A_1x_m^1)\}$
$= \{d_1^p(x_m^1, A_nA_{n-1}\ldots A_2\beta_2), d_1^p(x_m^1, x_{m+1}^1), d_2^p(\beta_2, x_{m+1}^2)\}$

As m tend to infinity in (3.9) and F is continuous in 0 we get

$$d_1^p(A_nA_{n-1}\ldots A_2\beta_2, \beta_1) \le cd_1^p(\beta_1, A_nA_{n-1}\ldots A_2\beta_2)$$

So we get, $\quad A_nA_{n-1}\ldots A_2\beta_2 = \beta_1$

In same way, we obtain

$$A_1A_n\ldots A_4A_3\beta_3 = \beta_2\ , A_2A_1A_n\ldots A_5A_4\beta_4 = \beta_3\,,\ .\ .\ .\ .\ ,\ A_{n-1}A_{n-2}\ldots A_2A_1\beta_1 = \beta_n$$

Using (3.1), taking $x^1 = \beta_1, x^2 = x_m^2$ we get

$$\begin{aligned} d_1^p(x_m^1, A_nA_{n-1}\ldots A_2A_1\beta_1) &= d_1^p(A_nA_{n-1}\ldots A_2x_m^2, A_nA_{n-1}\ldots A_2A_1\beta_1) \\ &\le c\max M_1(\beta_1, x_m^2) + F(\min M_1(\beta_1, x_m^2)) \end{aligned}$$

Where, $M_1(\beta_1, x_m^2) = \{d_1^p(\beta_1, A_nA_{n-1}\ldots A_2x_m^2), d_1^p(\beta_1, A_nA_{n-1}\ldots A_2A_1\beta_1), d_2^p(x_m^2, A_1\beta_1)\}$
$= \{d_1^p(\beta_1, A_nA_{n-1}\ldots A_2x_m^1), d_1^p(\beta_1, A_nA_{n-1}\ldots A_2A_1\beta_1), d_2^p(x_m^2, A_1\beta_1)\}$

Now letting m tend to infinity we get

$$\begin{aligned} d_1^p(\beta_1, A_nA_{n-1}\ldots A_2A_1\beta_1) &\le c\max\{d_1^p(\beta_1,\beta_1), d_1^p(\beta_1, A_nA_{n-1}\ldots A_2A_1\beta_1), d_2^p(\beta_2, A_1\beta_1)\} \\ &= c\max\{d_1^p(\beta_1, A_nA_{n-1}\ldots A_2A_1\beta_1), d_2^p(\beta_2, A_1\beta_1)\} \end{aligned}$$

From which it follows, or

$$d_1^p(\beta_1, A_nA_{n-1}\ldots A_2A_1\beta_1) \le cd_1^p(\beta_1, A_nA_{n-1}\ldots A_2A_1\beta_1) \Leftrightarrow A_nA_{n-1}\ldots A_2A_1\beta_1 = \beta_1$$

Or

$$d_1^p(\beta_1, A_nA_{n-1}\ldots A_2A_1\beta_1) \le cd_2^p(\beta_2, A_1\beta_1)$$

This can be also written in following form

$$d_1^p(\beta_1, A_n\beta_n) \le c d_2^p(\beta_2, A_1\beta_1) \tag{3.10}$$

Since, $A_{n-1}A_{n-2}\dots A_2A_1\beta_1 = \beta_n$

Taking $x^3 = x_m^3, x^2 = \beta_2$ in inequality (3.2) in the same way as above, we obtain

$$d_2^p(\beta_2, A_1\beta_1) \le c d_3^p(\beta_3, A_2\beta_2) \tag{3.11}$$

Continuously like above, we get

$$d_n^p(\beta_n, A_{n-1}\beta_{n-1}) \le c d_1^p(\beta_1, A_n\beta_n) \tag{3.12}$$

By (3.10), (3.11) and (3.12), we obtain

$$d_1^p(\beta_1, A_n\beta_n) \le c d_2^p(\beta_2, A_1\beta_1) \le c^2 d_3^p(\beta_3, A_2\beta_2) \le \dots \le c^{n-1} d_n^p(\beta_n, A_{n-1}\beta_{n-1}) \le c^n d_1^p(\beta_1, A_n\beta_n)$$
$$\Rightarrow d_1^p(\beta_1, A_n\beta_n) \le c^n d_1^p(\beta_1, A_n\beta_n)$$
$$\Leftrightarrow A_n\beta_n = \beta_1 \text{ , since } 0 \le c < 1$$

Thus again $d_1^p(\beta_1, A_nA_{n-1}\dots A_2A_1\beta_1) = d_1^p(\beta_1, A_n\beta_n) = 0$

$$\Leftrightarrow A_nA_{n-1}\dots A_2A_1\beta_1 = \beta_1$$

So, we proved that β_1 is a fixed point of $A_nA_{n-1}\dots A_2A_1\beta_1$

In same way it can be shown that $A_1A_n\dots A_3A_2$ has a unique fixed point $\beta_2 \in X_2$, $A_2A_1\dots A_4A_3$ has a unique fixed point $\beta_3 \in X_3$ and so on $A_{n-1}A_{n-2}\dots A_1A_n$ has a unique fixed point $\beta_n \in X_n$.

Further, we also showed that $A_1(\beta_1) = \beta_2, A_2(\beta_2) = \beta_3, A_3(\beta_3) = \beta_4, \dots, A_{n-1}(\beta_{n-1}) = \beta_n, A_n(\beta_n) = \beta_1$

Now let assume now that $\beta_1^1 \in X_1$ is another fixed point of $A_nA_{n-1}\dots A_2A_1$, different from β_1.

Using (3.1), if we take $x^1 = \beta_1, x^2 = A_1\beta_1^1$, we get

$$d_1^p(\beta_1^1, \beta_1) = d_1^p(A_nA_{n-1}\dots A_2A_1\beta_1^1, A_nA_{n-1}\dots A_2A_1\beta_1)$$
$$\le c \max M_1(\beta_1, A_1\beta_1^1) + F(\min M_1(\beta_1, A_1\beta_1^1))$$

Where, $M_1(\beta_1, A_1\beta_1^1) = \{d_1^p(\beta_1, A_nA_{n-1}\dots A_2A_1\beta_1^1), d_1^p(\beta_1, A_nA_{n-1}\dots A_2A_1\beta_1), d_2^p(A_1\beta_1^1, A_1\beta_1)\}$
$$= \{d_1^p(\beta_1, \beta_1^1), d_1^p(\beta_1, \beta_1), d_2^p(A_1\beta_1^1, \beta_2)\}$$

From which it follows

$$d_1^p(\beta_1^1, \beta_1) \le c d_2^p(A_1\beta_1^1, \beta_2) \tag{3.13}$$

Taking $x^3 = \beta_3, x^2 = A_1\beta_1^1$ in inequality (3.2) we obtain

$$d_2^p(\beta_2, A_1\beta_1^1) = d_2^p(A_1A_n\dots A_4A_3\beta_3, A_1A_n\dots A_3A_2A_1\beta_1^1)$$
$$\le c \max M_2(A_1\beta_1^1, \beta_3) + F(\min M_2(A_1\beta_1^1, \beta_3))$$

Where, $M_2(A_1\beta_1^1,\beta_3) = \{d_2^p(A_1\beta_1^1, A_1A_n\ldots A_3\beta_3), d_2^p(A_1\beta_1^1, A_1A_n\ldots A_2A_1\beta_1^1), d_3^p(\beta_3, A_2A_1\beta_1^1)\}$
$= d_2^p(A_1\beta_1^1,\beta_2), d_2^p(A_1\beta_1^1, A_1\beta_1^1), d_3^p(\beta_3, A_2A_1\beta_1^1)\}$

Then, $$d_2^p(\beta_2, A_1\beta_1^1) \le c d_3^p(\beta_3, A_2A_1\beta_1^1) \tag{3.14}$$

Continuously like above, using (3.4), we get

$$d_n^p(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1^1,\beta_n) \le c d_1^p(\beta_1^1,\beta_1) \tag{3.15}$$

By using (3.13), (3.14) and (3.15)

$$d_1^p(\beta_1^1,\beta_1) \le c d_2^p(A_1\beta_1^1,\beta_2) \le c^2 d_3^p(\beta_3, A_2A_1\beta_1^1) \le .. \le c^{n-1} d_n^p(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1^1,\beta_n) \le c^n d_1^p(\beta_1^1,\beta_1)$$
$$\Leftrightarrow d_1^p(\beta_1^1,\beta_1) \le c^n d_1^p(\beta_1^1,\beta_1) \text{ where } 0 \le c < 1.$$

It follows that $\beta_1^1 = \beta_1$.

Thus we proved β_1 is the fixed point of $A_nA_{n-1}\ldots A_2A_1$.

In the same way, it can be shown that $A_1A_n\ldots A_3A_2$ has a unique fixed point $\beta_2 \in X_2$, $A_2A_1\ldots A_4A_3$ has a unique fixed point $\beta_3 \in X_3$ and so on $A_{n-1}A_{n-2}\ldots A_1A_n$ has a unique fixed point $\beta_n \in X_n$.

Corollary3.2: if we take n=3 ,then we obtain [1] proved by Luljeta kikina.

Corollary3.3: if we take n=2 ,then we obtain [2] proved by S.Č.Nešić.

Corollary3.4: if we take n=3, p=2 and F(t)=0 for all t belongs to R^+, then we obtain a gerernalization of theorem 2[5], extended to three metric spaces.

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