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FRACTIONAL TWIN DOMINATION NUMBER IN DIGRAPHS

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ABSTRACT

Let D = (V, A) be any digraph. A function f: V → [0, 1] is called a twin dominating function (TDF) if the sum of its function values over any closed out-neighborhood is at least one as well as the sum of its function values over any closed in-neighborhood is at least one. A TDF f of D is called a minimal TDF if there is no TDF g of D such that $g(v) \le f(v)$ for all $v \in V$ and $g(v_0) \ne f(v_0)$ for some $v_0 \in V$. The weight of the function f is $|f| = \sum_{v\in V} f(v)$. The fractional twin domination number of D, denoted γ_{ft}(D), equals the minimum weight of a TDF of D. In this research paper, we proved a necessary and sufficient condition for a TDF to be minimal and found fractional twin domination number for the digraph namely directed cycle. We established some bounds of fractional twin domination number for any digraph and proved that these bounds were sharp. We also proved a necessary and sufficient condition for fractional twin domination number to be n, where there are n vertices of either outdegree 0 or indegree 0.

Keywords: *Digraphs, out-dominating function, in-dominating function, twin dominating function, minimal twin dominating function, fractional twin domination number γ_{ft}(D).*

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1. INTRODUCTION

We use [3] for notation and terminology which are not defined here. The concept of dominating function and fractional domination number have been introduced in [6]. A dominating function (DF) of a graph $G = (V, E)$ is a function $f : V \to [0,1]$ such that $\sum_{u\in N[v]} f(u) \ge 1$ for all $v \in V$,

where $N[v] = \{u \in V/u \text{ is adjacent with } v\} \cup \{v\}$. A DF f is called minimal dominating function (MDF) if there is no DF g of G such that $g(v) \le f(v)$ for all $v \in V$ and $g(v_0) \ne f(v_0)$ for some $v_0 \in V$. For a real valued function $f : V \to$ R, the weight of f is $|f| = \sum_{v\in V} f(v)$ and $S \subseteq V,\ f(S) = \sum_{v\in S} f(v)$ so $|f| = f(V)$. The boundary set B_f and the positive set P_f of a DF f are defined by $B_f = \{v \in V/f(N[v]) = 1\}$ and $P_f = \{v \in V/f(v) > 0\}$.

Let A and B be the subsets of V in which A dominates B and it can be written as $A \to B$, if every vertex in $B \setminus A$ is adjacent to some vertex in A. In this connection, the following theorem (Theorem 1.1) provides a necessary and sufficient condition for a DF to be an MDF.

Theorem 1.1: [2] A DF f of G is a MDF if and only if $B_f \to P_f$

For any graph G, the fractional domination number γ_f(G) is defined by γ_f(G) = min $\{|f|/f$ is a MDF of G$\}$. Although domination and other related concepts have been extensively studied for undirected graphs, the respective analogue on digraphs have not received much attention. Of course, a survey of results on domination in directed graphs by Ghoshal, Lasker and Pillone is found in chapter 15 of Haynes et al.[4], but most of the results in this survey chapter deals with the concepts of kernels and solutions in digraphs and also on dominations in tournaments. For a survey of dominating functions, we also refer the research reviews by Haynes et al [5]. In [7], we transferred the concept of

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dominating function (DF) and fractional domination number γ_f(G) to digraphs, which were called as out-dominating function and fractional out domination number γ_{fo}(D), and subsequently initiated a study pertaining to these concepts.

2. TWIN DOMINATING FUNCTION

A directed graph or digraph D consists of a finite nonempty set V together with a prescribed collection A of ordered pairs of distinct elements of V. The elements of V are called vertices and the elements of A are called arcs. If $(u,v) \in V$, we say that u is adjacent to v and v is adjacent from u. The outdegree od(v) of a vertex v is the number of vertices that are adjacent from it and the indegree id(v) is the number of vertices adjacent to it . Let $N^+(v)$ denote the set of all vertices of D which are adjacent from v. Let $N^+[v]$=$N^+(v)$∪$\{v\}$. Let $N^-(v)$ denote the set of all vertices of D which are adjacent to v. Let $N^-[v]$= $N^-(v)$∪$\{v\}$. The minimum indegree and the maximum indegree among the vertices of D are denoted as δ^- and Δ^- respectively. Similarly, the minimum outdegree and the maximum outdegree among the vertices of D are denoted as δ^+ and Δ^+ respectively.

Definition 2.1: [7] An out-dominating function (ODF) of a digraph D = (V,A) is a function $f : V \to [0,1]$ such that $\sum\limits_{u\in N^+[v]} f(u) \geq 1$ for all $v \in V$.

Definition 2.2: An in-dominating function (IDF) of a digraph D = (V,A) is a function $f : V \to [0,1]$ such that $\sum\limits_{u\in N^-[v]} f(u) \geq 1$ for all $v \in V$.

Definition 2.3: A twin dominating function (TDF) of a digraph D = (V,A) is a function $f : V \to [0,1]$ such that f is an out-dominating function as well as in-dominating function.

Definition 2.4: A TDF f is called minimal TDF if there is no TDF g of D such that $g(v) \leq f(v)$ for all $v \in V$ and $g(v_0) \neq f(v_0)$ for some $v_0 \in V$.

Definition 2.5: The fractional twin domination number γ_{ft}(D) is defined by γ_{ft}(D) = min {$|f|/f$ is a minimal TDF of D}.

Definition 2.6: [7] Let A and B be two subsets of V. We say that A out-dominates B and write $A \xrightarrow{out} B$ if every vertex $u \in B \setminus A$ is adjacent from some vertex in A.

Notation 2.7:
1. $B_f^+ = \{v \in V/f(N^+[v]) = 1\}$
2. $B_f^- = \{v \in V/f(N^-[v]) = 1\}$
3. $P_f = \{v \in V/f(v) > 0)$

Theorem 2.8: [7] An out-dominating function f of D is a minimal out dominating function if and only if $B_f^+ \xrightarrow{out} P_f$

Definition 2.9: Let A, B and C be three subsets of V. We say that A and B twin dominate C and write $(A,B) \to C$ if every vertex $u \in C \setminus (A \cup B)$ is either adjacent from a vertex in A or adjacent to a vertex in B .

Theorem 2.10: A twin dominating function f of a digraph D is a minimal twin dominating function if and only if $(B_f^+, B_f^-) \to P_f$

Proof: Assume that f is a minimal twin dominating function of D. If $P_f \setminus (B_f^+ \cup B_f^-) = \phi$, there is noting to prove. If $P_f \setminus (B_f^+ \cup B_f^-) \neq \phi$, let $v \in P_f \setminus (B_f^+ \cup B_f^-)$.

Therefore, $f(v) > 0,\ f(N^+[v]) > 1,\ f(N^-[v]) > 1.$

Claim: id (v) ≥ 1 and od(v) ≥ 1
Suppose not, then either id(v) = 0 or od(v) = 0. If id(v) = 0, $N^-[v] = \{v\}$. So, $f(N^-[v]) = f(v) \leq 1$, which is a contradiction (since $f(N^-[v]) > 1$). If od(v) = 0, a similar contradiction is obtained. Hence the claim.

We have to prove that v is either adjacent from a vertex in B_f^+ or adjacent to a vertex in B_f^-. Suppose not, vertices adjacent to v are not in B_f^+ and vertices adjacent from v are not in B_f^-. Let $v_1, v_2, \ldots, v_n$ be the vertices adjacent to v and $u_1, u_2, \ldots, u_m$ be the vertices adjacent from v.

Therefore, $f(N^+[v_1]) > 1,\ f(N^+[v_2]) > 1, \ldots, f(N^+[v_n]) > 1$ and $f(N^-[u_1] > 1$,
$f(N^-[u_2] > 1, \ldots, f(N^-[u_m] > 1$. Let $f(N^+[v]) = 1 + s$, $f(N^+[v_1]) = 1 + s_1$
$f(N^+[v_2]) = 1 + s_2, \ldots, f(N^+[v_n]) = 1 + s_n$,$f(N^-[v]) = 1 + t$,$f(N^-[u_1]) = 1 + t_1$,
$f(N^-[u_2]) = 1 + t_2, \ldots, f(N^-[u_m]) = 1 + t_m$, where $s, s_1, s_2, \ldots, s_n, t, t_1, t_2, \ldots, t_m$ are all positive real numbers. Let x = min $(s, s_1, s_2, \ldots, s_n, t, t_1, t_2, \ldots, t_m, f(v))$.

Define $g : V \to [0,1]$ by $g(u) = \begin{cases} f(u), & u \neq v \\ f(v) - x, & u = v \end{cases}$.

Then, $g(N^+[v]) = f(N^+[v]) - x = 1 + s - x \geq 1$ (since $s - x \geq 0)$
$g(N^+[v_i]) = f(N^+[v_i]) - x = 1 + s_i - x \geq 1$ (since $s_i - x \geq 0$), $i = 1, 2, \ldots, n$
$g(N^-[v]) = f(N^-[v]) - x = 1 + t - x \geq 1$ (since $t - x \geq 0)$
$g(N^-[u_i]) = f(N^-[u_i]) - x = 1 + t_i - x \geq 1$ (since $t_i - x \geq 0$), $i = 1, 2, \ldots, m$.

Also, $g(N^+[u]) = f(N^+[u]) \geq 1$ and $g(N^-[u]) = f(N^-[u]) \geq 1$ for all $u \in V \setminus \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m\}$. So g is a twin dominating function of D and, $g(u) \leq f(u)$ for all $u \in V$ and $f(v) \neq g(v)$. This is a contradiction as f is a minimal twin dominating function of D. Therefore, $(B_f^+, B_f^-) \to P_f$.

Conversely, assume that $(B_f^+, B_f^-) \to P_f$. Suppose f is not minimal. So there exists a twin dominating function g of *G* such that $g(u) \leq f(u)$ for all $u \in V$ and $g(u_0) \neq f(u_0)$ for some $u_0 \in V$. This implies that $f(u_0) > 0$ (since $g(u_0) < f(u_0)$) and hence $u_0 \in P_f$.

Case (i): $f(N^+[u_0]) = 1$

Then, $g(N^+[u_0]) < 1$ as $g(u) \leq f(u)$ for all $u \in V$ and $g(u_0) \neq f(u_0)$. This is a contradiction.

Case (ii): $f(N^-[u_0]) = 1$
This is similar to case (i)

Case (iii): $f(N^+[u_0]) > 1$ and $f(N^-[u_0]) > 1$. Therefore, $u_0 \in P_f \setminus (B_f^+ \cup B_f^-)$. Since $(B_f^+, B_f^-) \to P_f$, u_0 is either adjacent from a vertex v_0 in B_f^+ or adjacent to a vertex x_0 in B_f^-. If u_0 is adjacent from $v_0 \in B_f^+$, then $f(N^+[v_0]) = 1$ and hence $g(N^+[v_0]) < 1$ (since $g(u) \leq f(u)$ for all $u \in V$ and $g(u_0) \neq f(u_0)$). This is a contradiction as g is a twin dominating function. Similarly, if u_0 is adjacent to $x_0 \in B_f^-$, we get a contradiction. Therefore, f is a minimal twin dominating function.

Definition 2.11: Directed Cycle $\overrightarrow{C_n}$ is the digraph (V, A), where $V = \{v_1, v_2, \ldots, v_n\}$ and
$A = \{(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1)\}$, $n \geq 2$

Theorem 2.12: For $n \geq 2$, $\gamma_{ft}(\overrightarrow{C_n}) = \frac{n}{2}$

Proof: Let $\overrightarrow{C_n} = (V, A)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and $A = \{(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1)\}$. Define $f : V \to [0,1]$ by $f(v_i) = \frac{1}{2}$, $i = 1, 2, \ldots, n$. Clearly f is a twin dominating function of D. So, $\gamma_{ft}(\overrightarrow{C_n}) \leq |f| = \frac{n}{2}$. To prove $\gamma_{ft}(\overrightarrow{C_n}) \geq \frac{n}{2}$, let f be any twin dominating function of D. Then, $f(N^+[v_i]) \geq 1$ and $f(N^-[v_i]) \geq 1$,$i = 1, 2, \ldots, n$. (i.e) $f(v_1) + f(v_2) \geq 1$, $f(v_2) + f(v_3) \geq 1$,. . ., $f(v_n) + f(v_1) \geq 1$ and $f(v_1) + f(v_n) \geq 1$, $f(v_2) + f(v_1) \geq 1$,. . ., $f(v_n) + f(v_{n-1}) \geq 1$. Here, there are only n inequalities that are added up by which we get $2|f| \geq n$. (i.e) $|f| \geq \frac{n}{2}$ for any twin dominating function f of $\overrightarrow{C_n}$. Taking minimum over f on both sides of this inequality, we get $\gamma_{ft}(\overrightarrow{C_n}) \geq \frac{n}{2}$.

Theorem 2.13: For a digraph D = (V, A) with n vertices, max $\left(\frac{n}{\Delta^- + 1}, \frac{n}{\Delta^+ + 1}\right) \leq \gamma_{ft}$(D). Also the bound is sharp.

Proof: Let f be any twin dominating function of D. So $\sum_{v\in V} f(N^+[v])$ = $\sum_{v\in V} f(v)(id(v)+1)$ (since $f(v)$ repeats id(v) + 1 times in $\sum_{v\in V} f(N^+[v])$). (i.e.) $|V| \leq \sum_{v\in V} f(v)id(v) + |f|$ (since $f(N^+[v]) \geq 1$).

Therefore, $n \leq \sum_{v\in V} f(v)\Delta^- + |f|$ (since id(v)$\leq \Delta^-$)

(i.e.) $$\frac{n}{\Delta^+ + 1} \leq |f| \qquad (1)$$

Also, $\sum_{v\in V} f(N^-[v])$ = $\sum_{v\in V} f(v)(od(v)+1)$. Therefore, $|V| \leq \sum_{v\in V} f(v)od(v) + |f|$

(i.e.) $$\frac{n}{\Delta^- + 1} \leq |f| \qquad (2)$$

From (1) and (2), we get max $\left(\frac{n}{\Delta^- + 1}, \frac{n}{\Delta^+ + 1}\right) \leq |f|$. Taking minimum over f on both sides of this inequality,

we get max$\left(\frac{n}{\Delta^- + 1}, \frac{n}{\Delta^+ + 1}\right) \leq \gamma_{ft}$(D)

For the directed cycle $\overrightarrow{C_n}$, $\Delta^+ = \Delta^- = 1$.

Therefore, max $\left(\frac{n}{\Delta^- + 1}, \frac{n}{\Delta^+ + 1}\right) = \frac{n}{2}$. Also by theorem 2.12, $\gamma_{ft}(\overrightarrow{C_n}) = \frac{n}{2}$. So the bound is sharp.

Theorem 2.14: For a digraph D = (V, A) with n vertices, γ_{ft}(D) ≤ $\frac{n}{\delta' + 1}$ where δ'=min (δ^+, δ^-). Also the bound is sharp.

Proof: Define $f : V \to [0, 1]$ by $f(v) = \frac{1}{\delta' + 1}$ for all $v \in V$. Now, $|N^+[v]|$ = $od(v) + 1 \geq \delta^+ + 1 \geq \delta' + 1$ for all $v \in V$. Therefore, $f(N^+[v]) = \frac{|N^+[v]|}{\delta' + 1} \geq 1$ for all $v \in V$.

Similarly, $|N^-[v]|$ = $id(v) + 1 \geq \delta^- + 1 \geq \delta' + 1$ for all $v \in V$. Therefore, $f(N^-[v]) = \frac{|N^-[v]|}{\delta' + 1} \geq 1$ for all $v \in V$. Hence f is a twin dominating function of D. (i.e.) γ_{ft}(D)$\leq |f| = \frac{n}{\delta' + 1}$. By theorem 2.12, $\gamma_{ft}(\overrightarrow{C_n})$= $\frac{n}{2}$. Also in $\overrightarrow{C_n}$ $\delta^+ = \delta^- = 1$.

Therefore, $\delta' = \min(\delta^+, \delta^-) = 1.$. (i.e.) $\frac{n}{\delta' + 1} = \frac{n}{2}$. So the bound is sharp.

Theorem 2.15: Let D = (V, A) be a digraph in which there are n vertices of either outdegree 0 or indegree 0. Any vertex with an outdegree of at least 1 and indegree of at least 1 is adjacent to a vertex of outdegree 0 and adjacent from a vertex of indegree 0 if and only if γ_{ft}(D) = n.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices with either outdegree 0 or indegree 0. Let $U = V \setminus \{v_1, v_2, \ldots, v_n\}$. Then any vertex in U has an outdegree of at least 1 and an indegree of at least 1.

Assume that any vertex with an outdegree of at least 1 and indegree of at least 1 is adjacent to a vertex of outdegree 0 and adjacent from a vertex of indegree 0. Let f be any twin dominating function of D. Since either $od(v_i)$= 0 or $id(v_i)$=0$f(v_i) = 1, i = 1, 2 \ldots, n$. Therefore, $|f| = \sum_{u \in U} f(u) + \sum_{i=1}^{n} f(v_i)$= $\sum_{u \in U} f(u) + n$. (i.e.), $|f| \geq n$ for any twin dominating function f of D (since $\sum_{u \in U} f(u) \geq 0$). So, γ_{ft}(D) $\geq n$. To prove γ_{ft}(D) $\leq n$, define $f : V \to [0, 1]$ by $f(v_i) = 1$, $i = 1, 2, \ldots, n$ and $f(u) = 0$ for all $u \in U$. Clearly, $f(N^+[v]) \geq 1$and $f(N^-[v]) \geq 1$ for all $v \in V$. Therefore, f is a twin dominating function of D. Hence, γ_{ft}(D)$\leq$ $|f| = \sum_{u \in U} f(u) + \sum_{i=1}^{n} f(v_i)$ =n.

Conversely, assume that γ_{ft}(D) =n. Let f be any twin dominating function of D. Since $od(v_i) = 0$ or $id(v_i) = 0$, $f(v_i) = 1,\ i = 1, 2, \ldots, n$.

So, $|f| = \sum_{u \in U} f(u) + \sum_{i=1}^{n} f(v_i)$

Therefore, $|f| = \sum_{u \in U} f(u) + n$ (3)

Suppose that some vertex in V, say, v_0, is not adjacent to any of the vertices $v_1, v_2, \ldots, v_n$. Therefore, $N^+[v_0] \subseteq U$. So, $\sum_{u \in U} f(u) \geq 1$. Taking minimum over $f$on both side of (3), let min $\{ \sum_{u \in U} f(u) / f$ is a twin dominating function of D} = $1 + x$, where $x \geq 0$. So (3) becomes γ_{ft}(D) =$1 + x + n$. (i.e.) $n = 1 + x + n$. This implies that $1 + x = 0$. This is a contradiction. Hence, every $u \in U$ is adjacent to a vertex of either outdegree 0 or indegree 0. A vertex can not be adjacent to a vertex of indegree 0. So every $u \in U$ is adjacent to a vertex of outdegree 0. Similarly, we can prove that every $u \in U$ is adjacent from a vertex of indegree 0. Hence, any vertex with an outdegree of at least one and indegree of at least one is adjacent to a vertex of outdegree 0 and adjacent from a vertex of indegree 0.

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