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CHARACTERIZATION OF LATTICE SIGMA ALGEBRAS

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ABSTRACT

This paper empowers some basic elementary properties of lattice algebra, lattice sigma algebra. Also it establishes that the countable intersection of lattice sigma algebras is again a lattice sigma algebra and the lattice sigma algebra generated by itself contains the collection of all unions of lattice measurable sets. Finally it ascertains some elementary properties of lattice measurable functions.

1. INTRODUCTION

The concept of lattice measurable functions was obtained by Anil kumar etrl (2011). The fundamentals of measure theoretical concepts were firstly described by Halmos (1974). Later on a detailed attempt has been made by Royden (1981). The concept of measure of a lattice has been effort by Szasz (1963). The perspective of the measure of a lattice has been attempted by G. Szasz (1963) and for signed lattice measure is originated by Tanaka (2009).

Section 2, deals with the fundamentals of lattice sigma algebra, lattice measure on a lattice sigma algebra with reference to Tanaka (2009). Further the concepts of lattice measurable set, lattice measure space and lattice σ – finite measure were defined based on Anil kumar etrl (2011). Here some basic elementary properties of lattice algebra and lattice sigma algebra were proved. Also it has been established that the countable intersection of lattice sigma algebras is again a lattice sigma algebra and the lattice sigma algebra generated by itself contains the collection of all unions of lattice measurable sets.

In section 3, the concepts of function lattice and lattice measurable function were initiated with allusion to Anil kumart etrl (2011). Here some elementary properties of lattice measurable functions have been established.

2. PRILIMINARIES

In this Paper, we shall consider the union and intersection notion of set theory as $\wedge$ and $\vee$ and we shall briefly review the well-known facts about lattice theory (Birkhoff [8]), propose an extension lattice, and investigate its properties.

$(L, \wedge, \vee)$ is called a lattice if it is enclosed under operations $\wedge$ and $\vee$ and satisfies, for any elements x, y, z, in L:
(L1) the commutative law: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.
(L2) the associative law: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$.
(L3) the absorption law: $x \vee (y \wedge x) = x$ and $x \wedge (y \vee x) = x$.

Hereafter, the lattice $(L, \wedge, \vee)$ will often be written as L for simplicity.

A mapping h from a lattice L to another lattice L is called a lattice-homomorphism, if it satisfiesh $(x \wedge y) = h(x) \wedge h(y)$ and $h(x \vee y) = h(x) \vee h(y)$, $\forall x, y \in L$.

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If h is a bijection, that is, h is one-to-one and onto, it is called a lattice isomorphism, and in this case, L^1 is said to be lattice-isomorphic to L.

A lattice $(L, \wedge, \vee)$ is called distributive if, for any x, y, z, in L.

(L4) the distributive law holds:

$x \vee (y \wedge z) = (x \vee y) \wedge (y \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (y \wedge z)$.

A lattice L is called complete if, for any subset A of L, L contains the supremum $\vee A$ and the infimum $\wedge A$. If L is complete, then L itself includes the maximum and minimum elements which are often denoted by 1 and 0 or I and O respectively [8].

A distributive lattice is called a Boolean lattice if for any element x in L, there exists a unique complement x^c such that

$x \vee x^c = 1$ (L5) the law of excluded middle

$x \wedge x^c = 0$ (L6) the law of non-contradiction

Let L be a lattice and $c: L \to L$ be an operator. Then € is called a lattice complement in L if the following conditions are satisfied.

(L5) and (L6); $\forall x \in L, x \vee x^c = 1$ and $x \wedge x^c = 0$,

(L7) the law of contrapositive; $\forall x, y \in L, x \leq y$ implies $x^c \geq y^c$,

(L8) the law of double negation; $\forall x \in L, (x^c)^c = x$.

Lattice Algebras

Unless otherwise stated, X is the entire set and S is a lattice of any subsets of X.

Definition 1: If S is a lattice and satisfies the following conditions, then it is called a lattice algebra.

(i) For all $h \in S, h^c \in S$

(ii) For all $a, b \in S, a \vee b \in S$.

Example 1: {∅, X} is a lattice algebra.

Example 2: Let $X = \Re$ and $S = \{E < \Re$ / E is finite or E^c is finite}. Here S is a lattice algebra.

Result 1: If $E_1, E_2 \in S$ then $E_1 \wedge E_2 \in S$.

Proof: If $E_1 \in S$, by the definition 1, $E_1^c \in S$ again $E_2 \in S$, by definition 1, $E_2^c \in S$.

Now $E_1^c, E_2^c \in S$, then $E_1^c \vee E_2^c \in S$. This implies $(E_1 \wedge E_2)^c \in S$. Hence $E_1 \wedge E_2 \in S$.

Result 2: If $E_1, E_2 \in S$ then $E_1 - E_2 \in S$

Proof: Let $E_2 \in S$ by definition $E_2^c \in S$

Now $E_1, E_2^c \in S$. Which implies $E_1 \wedge E_2^c \in S$ (by result 1).

Hence $E_1 - E_2 \in S$.

Theorem 1: If S is lattice algebra of subsets of X then

1. X∈S
2. ∅∈S

Proof: 1. Since S is nonempty, there exists A∈S.

Hence $A^c \in S$. So $X = A \vee A^c \in S$. Clearly $\emptyset = X^c \in S$.

Theorem 2: Suppose that S is lattice algebra of subsets of X and that $A_i \in S$ for each i in a finite index set I.

1. $\bigvee_{i\in I} A_i \in S$
2. $\bigwedge_{i\in I} A_i \in S$

Proof: We prove this theorem by using induction on the number of elements in I.

Let $x_1, x_2, \ldots\ldots x_n \in S$.

Since $x_1, x_2 \in S$ and S is a lattice, we have that $x_1 \wedge x_2, x_1 \vee x_2 \in S$.

Suppose the induction hypothesis that
$x_1 \wedge x_2 \wedge \ldots\ldots x_{n-1}, x_1 \vee x_2 \vee \ldots\ldots \vee x_{n-1} \in S$.

Since $x_1 \wedge x_2 \wedge \ldots\ldots x_{n-1} \in S$, $x_n \in S$ and S is a lattice,

We have $x_1 \wedge x_2 \ldots\ldots \wedge x_{n-1} \wedge x_n \in S$.

Since $x_1 \vee x_2 \vee \ldots\ldots \vee x_{n-1} \in S$, $x_n \in S$ and S is a lattice

We have that $x_1 \vee x_2 \vee \ldots\ldots \vee x_{n-1} \vee x_n \in S$.

It is clear that $x_1 \wedge x_2 \ldots\ldots .. \wedge x_n$ is the $\inf\{x_1, x_2, \ldots\ldots x_n\}$ and $x_1 \vee x_2 \vee \ldots.. x_n$ is the $\sup\{x_1, x_2, \ldots\ldots x_n\}$.

This theorem is true for all positive integers n

Therefore $\bigvee_{i\in I} A_i \in S$ and $\bigwedge_{i\in I} A_i \in S$.

Lattice σ-Algebras:

Unless otherwise stated, X is the entire set and S is a lattice of any subsets of X.

Definition 2: [7] If a lattice S satisfies the following conditions, then it is called a lattice σ - algebra;

(1) For all $h \in S$, $h^c \in S$.

(2) For all $h_n \in S$ for n = 1, 2, 3 ....., then $\bigvee_{n=1}^{\infty} h_n \in S$.

Example 3:
1. $\{\emptyset, X\}$ is a lattice σ -algebra.
2. P(X) power set of X is a lattice σ –algebra.

Example 4: [4] Let $X = \Re$, S = {Lebesgue measurable subsets of $\Re$} with usual ordering (≤). Here S is a lattice σ - algebra.

Example 5: Suppose S = {all topologies on X}. Here S is a complete lattice but not σ –algebra
.

Example 6: Let $X = \Re$ and $S = \{E < \Re$ / E is finite or E^c is finite}. Here S is lattice algebra but not lattice σ -algebra.

Note 1: Lattice σ-algebra of subsets of X is closed under countable unions and intersections.

Definition 3: If $m: S \to R \vee \{\infty\}$ satisfies the following properties, then m is called a lattice measure on the lattice σ -algebra S.

(1) $m(\emptyset) = m(0) = 0$.
(2) for all $h, g \in S$ such that $m(h), m(g) \geq 0$; $h \leq g \Rightarrow m(h) \leq m(g)$.
(3) for all $h, g \in S$: $m(h \vee g) + m(h \wedge g) = m(h) + m(g)$.
(4) If $h_n \in S$, $n \in N$ such that $h_1 \leq h_2 \leq \ldots \leq h_n \leq \ldots.$, then $m(\bigvee_{n=1}^{\infty} h_n) = \lim m(h_n)$.

Let m_1 and m_2 be lattice measures define on the same lattice σ -algebra S. If one of them is finite, the set function m (E) = m_1(E) - m_2(E), E $\in$ S is well defined and countably additive on S. However, it is necessarily nonnegative; it is called a signed lattice measure.

Example 7: Let X be any set S = P(X) be the class of all sub sets of X. Define for any A $\in$ S, m(A) = $+\infty$ if A is infinite = |A| if A is finite. Where |A| is the number of elements in A. Then m is a countable additive set function defined on S and hence m is a lattice measure on S.

Definition 4: [1] The ordered pair (X, S) is said to be lattice measurable space.

Example 8: X = $\Re$, S = {All Lebesgue measurable sub sets of $\Re$ }, ($\Re$, S) is a lattice measurable space.

Definition 5: [1] A set A is said to be lattice measurable set or lattice measurable if A belongs to S.

Example 9: [5] The interval (a, ∞) is a lattice measurable under usual ordering.

Example 10: [6] [0, 1] < $\Re$ is lattice measurable under usual ordering.

Example 11: [5] Every Borel lattice is a lattice measurable.

Theorem 3: If $A_i \in$ S for each i in a countable index set I, then $\underset{i\in I}{\wedge} A_i \in$ S.

Proof: We prove this theorem by using theorem 2. If $A_i \in$ S for i ∈ I then $A_i^c \in S$ for i ∈ I.

Therefore $\underset{i\in I}{\vee} A_i^c \in S$ and hence $\underset{i\in I}{\wedge} A_i = (\underset{i\in I}{\vee} A_i^c)^c \in S$.

Theorem 4: Suppose that S is a set and that S is a finite lattice algebra of subsets of X. Then S is also a lattice σ-algebra.

Proof: We prove this theorem by using theorem 2.

We just add a lot of empty sets, that is

$A_1 \vee A_2 = A_1 \vee A_2 \vee \emptyset \vee \emptyset \vee \ldots\ldots\ldots$.

Now we have an infinite sub sets.

Note 2: However, there are lattice algebras that are not lattice σ-algebras.

Theorem 5: The collection of finite and co-finite subsets of N defined below is a lattice algebra of subsets of N, but not a lattice σ-algebra:

F= {A ⊆ N:A is finite or A^c is finite}

Proof: Naturally N ∈ F since $N^c = \emptyset$ is finite. If A ∈ F then A^c ∈ F by the symmetry of the definition lattice algebra. Suppose that A, B ∈ F. If A and B are both finite then $A \vee B$ is finite. If A^c or B^c is finite, then $(A \vee B)^c = A^c \wedge B^c$ is finite. In either case, $A \vee B \in$ F. Thus F is a lattice algebra of subsets of N.

Let A_n={2n} for n ∈ N. Then A_n is finite, and hence $A_n \in$ F for each n ∈ N.

Let E= $\overset{\infty}{\underset{n=0}{\vee}} A_n$ Note that E and E^c are infinite, so E ∉ F. Thus F is not a lattice σ-algebra.

Note 3: P(X) denotes the collection of all subsets of X, called the power set of X. Trivially, P(X) is the largest lattice σ-algebra of X, at the other extreme, the smallest lattice σ-algebra of X is the collection {∅, X}.

Theorem 6: Suppose that S_i is a lattice σ-algebras of subsets of X for each i in a nonempty index set I. Then $S=\underset{i\in I}{\wedge} S_i$ also a lattice σ-algebra of subsets of X.

Proof: Let $A \in S$ then $A \in S_i$ for each $i \in I$ and hence $A^c \in S_i$ for each $i \in I$. Therefore $A^c \in S$. Suppose that $A_j \in S$ for each j in a countable index set J. Then $A_j \in S_i$ for each $i \in I$ and $j \in J$ and therefore $\underset{j\in J}{\vee} A_j \in S_i$ for each $i \in I$. It follows that $\underset{j\in J}{\vee} A_j \in S$.

Note 4: Suppose that B is a collection of subsets of X, in general B will not be a lattice σ-algebra. The lattice σ-algebra generated by B is the intersection of all lattice σ-algebras that contains B, it is denoted by σ (B) = $\wedge$ {S: S is a lattice σ-algebra of subsets of X and B⊆S}

Note 5: The collection of lattice σ-algebras in the intersection is non empty, since P(X) is in the collection.

Theorem 7: If A is a subset of X then σ {A} = {∅, A, A^c, X}

Proof: Let us consider the collection of subsets A= {A_i: i ∈ I} is a partition of X, if $A_i \wedge A_j = \emptyset$ for i, j ∈ I with i ≠ j, and $\underset{i\in I}{\vee} A_i = X$.

Theorem 8: Suppose that A={A_i : i ∈ I} is a countable partition of X. Then σ(A) is the collection of all unions of sets in A. That is, σ(A)={ $\underset{j\in J}{\vee} A_j$: J⊆I }.

Proof: Let S={ $\underset{j\in J}{\vee} A_j$: J⊆I }. Clearly X ∈ S (since $X = \underset{i\in I}{\vee} A_i$).

Suppose that B ∈ S. Then $B = \underset{j\in J}{\vee} A_j$ for some J⊆I. But then $B^c = \underset{j\in J^c}{\vee} A_j$, so $B^c \in S$.

Again suppose that $B_k \in S$ for $k \in K$ where K is a countable index set. Then for each $k \in K$ there exists $J_k \subseteq I$ such that $B_k = \underset{j\in J_k}{\vee} A_j$. But then $\underset{k\in K}{\vee} B_k = \underset{k\in K}{\vee}\ \underset{j\in J_k}{\vee} A_j = \underset{j\in J}{\vee} A_j$ where $J = \underset{k\in K}{\vee} J_k$. Hence $\underset{k\in K}{\vee} B_k \in S$. Therefore S is a lattice σ-algebra of subsets of X. Trivially, A⊆S. If T is a lattice σ-algebra of subsets of X and A ⊆ T, then clearly $\underset{j\in J}{\vee} A_j \in T$ for every J ⊆ I. Hence S ⊆ T.

If $A_i \neq \emptyset$ for i ∈ I then the unions in σ(A) are distinct. That is, if J, K ⊆ I and J ≠ K then $\underset{j\in J}{\vee} A_j \neq \underset{k\in K}{\vee} A_k$. In particular, if there are n nonempty sets in A, then there are 2^n subsets of I and hence 2^n sets in σ(A).

3. Lattice Measurable Functions

Definition 6: [2] A function lattice is a collection S^1 of extended real valued functions defined on a set S which is a lattice with respect to usual partial ordering on functions. That is if $f,g \in S^1$ then $f \vee g \in S^1$, $f \wedge g \in S^1$.

Example 12: Let f: $\mathfrak{R} \to \mathfrak{R}$ the set of all real valued functions defined on $\mathfrak{R}$. Then $\mathfrak{R}$ under usual ordering is a function lattice.

Definition 7: If f and g are extended real valued lattice measurable functions defined on S^1, then $f \vee g$, $f \wedge g$ are defined by

$(f \vee g)(x) = \max\{f(x), g(x)\}$ and
$(f \wedge g)(x) = \min\{f(x), g(x)\}$ for any $x \in S$.

Note 6: If g(x) = 0 for all x, then
$(f \vee 0)(x) = f(x)$ if $f(x) \geq 0$, 0 if $f(x) < 0$ and
$(f \wedge 0)(x) = 0$ if $f(x) \geq 0$, $f(x)$ if $f(x) < 0$.

Definition 8: [2] Lattice measurable function: An extended real value function f defined on a lattice measurable E is said to be lattice measurable function if the set $\{x \in E / f(x) > \alpha\}$ is lattice measurable for all real numbers α.

Example 13:
(1) Constant functions are lattice measurable functions.
(2) If A is a lattice measurable function in $\Re$, then χ_A is lattice measurable function.
(3) Continuous functions from $\Re \rightarrow \Re$ are lattice measurable functions.
(4) Any step function is lattice measurable function.

Theorem 9: Suppose that R, S, and T are sets with lattice σ-algebras R, S, and T, respectively. If f: R→S is lattice measurable and g : S→T is lattice measurable, then g∘f : R→T is lattice measurable.

Proof: If A ∈ T then $g^{-1}(A) \in S$ since g is lattice measurable, and hence $(g \circ f)^{-1}(A) = f^{-1}[g^{-1}(A)] \in R$ since f is lattice measurable. If T is smallest possible lattice σ-algebra or if S is largest one, then any function from S into T is lattice measurable.

Theorem 10: If T= {∅, T} or if a lattice σ-algebra S = P(S) then every lattice measurable function f: S→T is lattice measurable.

Proof: Let us consider a lattice measurable function f: S→T. Suppose that T= {∅, T} and that S is an arbitrary lattice σ-algebra on S. The $f^{-1}(T) = S \in S$ and $f^{-1}(\emptyset) = \emptyset \in S$ so f is lattice measurable. Next suppose that S = P(S) and that T is an arbitrary lattice σ-algebra on T. Then trivially $f^{-1}(A) \in S$ for every A ∈ T so again f is lattice measurable.

Theorem 11: Suppose that f : S→T be a lattice measurable function , and T is a lattice σ-algebra of subsets of S. The collection $\sigma(f) = \{f^{-1}(A) / A \in T\}$ is a lattice σ-algebra of subsets of S, called the lattice σ-algebra generated by f.

Proof: Let S ∈ σ(f) since *T* ∈ T and $f^{-1}(T)=S$. If *B* ∈ σ(f) then $B = f^{-1}(A)$ for some A ∈ T. But then $A^c \in T$ and hence $B^c = f^{-1}(A^c) \in \sigma(f)$.

Suppose that $B_i \in \sigma(f)$ for i, in a countable index set I. Then for each i ∈ I there exists $A_i \in T$ such that $B_i = f^{-1}(A_i)$. But then $\vee_{i\in I} A_i \in T$ and $\vee_{i\in I} B_i = f^{-1}(\vee_{i\in I} A_i)$. Hence $\vee_{i\in I} B_i \in \sigma(f)$.

The lattice σ-algebra generated by f is the smallest lattice σ-algebra on S that makes f measurable (relative to the given lattice σ-algebra on T). More generally, suppose that T_i is a set with lattice σ-algebra T_i for each i in a nonempty index set I, and that $f_i : S \rightarrow T_i$ for each i∈I. The lattice σ-algebra generated by this collection of functions is

$\sigma\{f_i / i \in I\} = \sigma\{f^{-1}(A) / i \in I, A \in T_i\}$

Again, this is the smallest lattice σ-algebra on S that makes f_i measurable for each i∈I.

Theorem 12: Suppose that S has lattice σ-algebras R and S with R⊆S, and that T has lattice σ-algebras U and T with U⊆T. If f:S→T is lattice measurable with respect to R and T, then f is lattice measureable with respect to S and U.

Proof: If A∈U then A∈T. Hence $f^{-1}(A) \in R$ so $f^{-1}(A) \in S$.

CONCLUSION:

This manuscript confirms some basic elementary properties of lattice algebras and lattice sigma algebras. Also it was established that the countable intersection of lattice sigma algebras is again lattice sigma algebra and the lattice sigma algebra generated by itself contains the collection of all unions of lattice measurable sets. It was proved that the composition of two lattice measurable functions is lattice measurable. It has been identified the concept of σ-algebra generated by lattice measurable function. Finally it confirms some elementary properties of lattice measurable functions.

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