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ON LOWER AND UPPER SEMI-QUASICONTINUOUS FUNCTIONS

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ABSTRACT

In this paper, some interesting properties of lower and upper semi-quasicontinuous functions are investigated. The relationship of these functions with other well-known notions like lower (upper) semicontinuity are established.

Key words: *Lower semi-quasicontinuity, Upper semi-quasicontinuity, Lower semicontinuity, Upper semicontinuity, Semi-continuity.*

AMS subject classification: *54C08, 54C30.*

INTRODUCTION

The notion of lower (upper) semi-quasicontinuity is similar to the well-known notion of lower (upper) semicontinuity. Some properties are similar but there are remarkable differences which are investigated and studied in this paper. Examples are provided wherever necessary. It is also proved that the set of all real valued lower (upper) semi-quasicontinuous functions defined on a topological space is closed under uniform limits.

In what follows X and $\mathbb{R}$ stand for a topological space and the real line respectively. The symbols $cl(\ )$ and $Int(\ )$ denote the closure and the interior in X respectively. The symbol ξ_A represents the characteristic function of a subset A of X.

1. PRELIMINARIES

The Difinitions that are needed throughout this paper are presented in this section.

1.1 Definition [2]: A function $f: X \to \mathbb{R}$ is said to be lower semi-quasicontinuous ($lsqc$) at a point $x_0 \in X$ if for every $\varepsilon > 0$ and every neighborhood U of x_0 in X there exists a non-empty open set $W \subset U$ such that $-\varepsilon < f(x) - f(x_0) \quad \forall\ x \in W$.

If f is $lsqc$ at every point of X then f is said to be *lsqc* on X. The real set of all valued $lsqc$ functions on X is denoted by $\mathcal{SC}^+(X)$.

1.2 Definitions [2]: A function $f: X \to \mathbb{R}$ is said to be upper semi-quasicontinuous ($usqc$) at a point $x_0 \in X$ if for every $\varepsilon > 0$ and every neighborhood U of x_0 in X there exists a non-empty open set $W \subset U$ such that $f(x) - f(x_0) < \varepsilon \quad \forall\ x \in W$

If f is $usqc$ at every point of X then f is $usqc$ on X. The set of all real valued $usqc$ functions on X is denoted by $\mathcal{SC}^-(X)$.

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1.3 Definition [5]: A function $f : X \to \mathbb{R}$ is said to be lower semicontinuous (lsc)at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists a neighborhood G of x_0 in X such that

$$-\varepsilon < f(x) - f(x_0) \qquad \forall \ x \in G \ .$$

If f is lsc at every point of X then f is lsc on X. The set of all real valued lsc functions on X is denoted by $\mathscr{C}^+(X)$.

1.4 Definition [5]: A function $f : X \to \mathbb{R}$ is said to be upper semicontinuous(usc) at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists a neighborhood G of x_0 in X such that $f(x) - f(x_0) < \varepsilon \quad \forall \ x \in G$.

If f is usc at every point of X then f is usc on X. The set of all real valued usc functions on X is denoted by $\mathscr{C}^-(X)$.

1.5 Definition [3]: A function $f : X \to \mathbb{R}$ is said to be semi-continuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists a neighborhood U of x_0 in X there exists a non-empty open set $W \subset U$ such that $|f(x) - f(x_0)| < \varepsilon \quad \forall \ x \in W$.

If f is semi-continuous at every point of X then it is said to be semi-continuous on X. *S*ome Authors call these functions Quasicontinuous (see [2]).

1.6 Definition [4]: A subset A of X is said to be semi-open if $A \subseteq cl(Int(A))$ or equivalently if there exists an open set O in X such that $O \subseteq A \subseteq cl(O)$.

1.7 Definition [1]: A subset A of X is said to be semi-closed if the complement $A^c = X - A$ of A is semi-open in X.

2. RELATIONSHIP AMONG *LSQC*, *USQC*, *LSC* AND *USC* FUNCTIONS

2.1 Proposition: If $f : X \to \mathbb{R}$ is semi-continuous at a point $x_0 \in X$ then f is both $lsqc$ and $usqc$ at x_0.

Proof: Let $f : X \to \mathbb{R}$ be semi-continuous at $x_0 \in X$. Let $\varepsilon > 0$ be given and let U be a neighborhood of x_0 in X .
Then there exists a non-empty open set $W \subset U$ such that

$$|f(x) - f(x_0)| < \varepsilon \ \ \forall \quad x \in W$$
$$\Rightarrow -\varepsilon < f(x) - f(x_0) < \varepsilon \quad \forall \ x \in W$$
$$\Rightarrow -\varepsilon < f(x) - f(x_0) \quad \text{and} \quad f(x) - f(x_0) < \varepsilon \ \ \forall \ x \in W$$
$$\Rightarrow f \text{ is both } lsqc \text{ and } usqc \text{ at } x_0 .$$

2.2 Remark: The converse of the above Proposition 2.1 is not true as is evident from the following example.

2.3 Example: Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad x = \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$$

Clearly f is both lsqc and usqc at $x_0 = \frac{1}{2}$, but it is not semi-continuous at $x_0 = \frac{1}{2}$.

2.4 Proposition: (a) $f \in \mathscr{C}^{+}(X) \Rightarrow f \in \mathscr{SC}^{+}(X)$
(b) $f \in \mathscr{C}^{-}(X) \Rightarrow f \in \mathscr{SC}^{-}(X)$

2.5 Remark: The converses of (a) and (b) in the Proposition 2.4 are not true. The function in the example 2.3 is both $lsqc$ and $usqc$ at $x_0 = \frac{1}{2}$. But for $0 < \varepsilon < \frac{1}{2}$ and for any neighborhood $U = (x_0 - \delta, x + \delta)$ of $x_0 = \frac{1}{2}$ in [0,1] for $\delta > 0$ and $x_0 < x < x_0 + \delta$ we have $f(x) - f(x_0) = \frac{1}{2} > \varepsilon \Rightarrow f$ is not *usc* at $x_0 = \frac{1}{2}$. It can be easily verified that f is not lsc at $x_0 = \frac{1}{2}$.

2.6 Proposition: $f \in \mathscr{SC}^{-}(X) \Leftrightarrow -f \in \mathscr{SC}^{+}(X)$

Proof: Let $x_0 \in X$. f is $usqc$ at x_0 $\Leftrightarrow$ for every $\varepsilon > 0$ and every neighborhood U of x_0 in X there exists a non-empty open set $W \subset U$ such that

$$f(x) - f(x_0) < \varepsilon \quad \forall \ x \in W$$

$$\Leftrightarrow (-f)(x) - (-f)(x_0) = -[f(x) - f(x_0)] > -\varepsilon \quad \forall \ x \in W$$

$\Leftrightarrow$ f is $lsqc$ at x_0.

Hence $f \in \mathscr{SC}^{-}(X) \Leftrightarrow -f \in \mathscr{SC}^{+}(X)$.

3. CHARACTERIZATIONS OF *LSQC* AND *USQC* FUNCTIONS

3.1 Proposition: Let $A \subset X$. Then

(a) $f \in \mathscr{SC}^{+}(X)$ if and only if for every $s \in \mathbb{R}$ the set $G_s = \{x \in X \ / \ f(x) > s\}$ is semi-open X.

(b) $f \in \mathscr{SC}^{-}(X)$ if and only if for every $s \in \mathbb{R}$ the set $H_s = \{x \in X \ / \ f(x) < s\}$ is semi-open X.

(c) $\xi_A \in \mathscr{SC}^{+}(X)$ if and only if A is semi-open X.

(d) $\xi_A \in \mathscr{SC}^{-}(X)$ if and only if A is semi-closed X.

Proof: (a) Suppose that $f \in \mathscr{SC}^{+}(X)$. Let $s \in \mathbb{R}$. Let $x_0 \in G_s$

$\Rightarrow f(x_0) > s$

$\Rightarrow f(x_0) - s > 0$

$\varepsilon = f(x_0) - s$. Let U be a neighborhood of x_0 in X.

Since f is $lsqc$ at x_0, there exists a non-empty open set $W \subset U$ such that $x \in W$

$\Rightarrow -\varepsilon < f(x) - f(x_0)$

$\Rightarrow -f(x_0) + s < f(x) - f(x_0)$

$\Rightarrow f(x) > s$

$\Rightarrow x \in G_s$

$\therefore W \subseteq G_s$ and hence $W \subset U \cap G_s$.

Thus for each $x_0 \in G_s$ there exists a non-empty open set $W(x_0) \subseteq U \bigcap G_s$ such that Put $O = \bigcup\limits_{x_0 \in G_s} W(x_0)$. Then O is a non-empty open set in $\mathcal{X}$ such that $O \subseteq G_s$.

Assume that $G_s \bigcap (cl(O))^c \neq \phi$
$\Rightarrow$ there exists a point $p \in \mathcal{X}$ such that $p \in G_s \bigcap (cl(O))^c$
$\Rightarrow$ $p \in G_s$ and $p \notin cl(O)$

Since $p \notin cl(O)$, there exists a neighborhood U of p in $\mathcal{X}$ such that $U \bigcap O = \phi$.

Since $p \in G_s$, there exists a non-empty open set $W(p)$ such that $W(p) \subseteq U \bigcap G_s$.

This is a contradiction.

$\therefore\ G_s \bigcap (cl(O))^c = \phi \quad \Rightarrow \quad G_s \subseteq cl(O)$

Thus there is a non-empty open set O in X such that $O \subseteq G_s \subseteq cl(O)$

$\Rightarrow$ G_s is semi-open in $\mathcal{X}$. Thus for every $s \in \mathbb{R}$ the set $G_s = \{x \in \mathcal{X} / f(x) > s\}$ is semi-open in $\mathcal{X}$.

Conversely suppose that for every $s \in \mathbb{R}$, the set $G_s = \{x \in \mathcal{X} / f(x) > s\}$ is semi-open in $\mathcal{X}$.

Now we prove that $f \in \mathcal{SC}^+(\mathcal{X})$.

Let $t \in \mathcal{X}$ and U be a neighborhood of t in $\mathcal{X}$. Let $\varepsilon > 0$ be given.

Put $s = f(t) - \varepsilon$. For this $s \in \mathbb{R}$, the set $G_s = \{x \in \mathcal{X} / f(x) > f(t) - \varepsilon\}$ is semi-open in $\mathcal{X}$.

Clearly $t \in G_s$ and hence $U \bigcap G_s$ is a semi-neighborhood of t in $\mathcal{X}$.

$\Rightarrow$ there exists a non-empty open set W such that $W \subseteq U \bigcap G_s \subseteq cl(W)$

$$\begin{aligned} \therefore \quad x \in W \quad &\Rightarrow \quad x \in U \bigcap G_s \\ &\Rightarrow \quad x \in U \text{ and } x \in G_s \\ &\Rightarrow \quad f(x) > f(t) - \varepsilon. \end{aligned}$$

Thus for every $\varepsilon > 0$ and every neighbohood U of t in X there exists a non- empty open set $W \subset U$ such that $-\varepsilon < f(x) - f(t) \quad \forall \ x \in W$

$\Rightarrow$ f is $lsqc$ at t. Thus f is $lsqc$ on $\mathcal{X}$. Hence $f \in \mathcal{SC}^+(\mathcal{X})$.

(b) Suppose that $f \in \mathcal{SC}^-(\mathcal{X})$
$\Leftrightarrow$ $-f \in \mathcal{SC}^+(\mathcal{X})$
$\Leftrightarrow$ the set $\{x \in \mathcal{X} \ / -f(x) > -s\}$ is semi-open for $s \in \mathbb{R}$
$\Leftrightarrow$ the set $H_s = \{x \in \mathcal{X} \ / f(x) < s\}$ is semi-open for every $s \in \mathbb{R}$.

(c) Suppose that $\xi_A \in \mathcal{SC}^+(\mathcal{X})$

for $s = \dfrac{1}{2}$, the set $G_s = \{x \in \mathcal{X} / \xi_A(x) > s\} = A$ is semi-open.

Conversely suppose that A is semi-open in $\mathcal{X}$.

Then $$G_s = \begin{cases} \phi & \text{if} \quad s \geq 1 \\ A & \text{if} \quad 0 \leq s < 1 \\ A \bigcup A^c & \text{if} \quad s < 0 \end{cases}$$

$\Rightarrow \quad G_s = \{x \in \mathcal{X} / \xi_A(x) > s\}$ is semi-open for every $s \in \mathbb{R}$.

$\Rightarrow \quad \xi_A \in \mathscr{SC}^+(\mathcal{X})$

(d) Suppose that $\xi_A \in \mathscr{SC}^-(\mathcal{X})$

$\Rightarrow \quad$ for $s = \dfrac{1}{2}$, the set $H_s = \{x \in \mathcal{X} / \xi_A(x) < s\} = A^c$ is semi-open.

$\Rightarrow \quad A$ is semi-closed in $\mathcal{X}$.

Conversely suppose that A is semi-closed in $\mathcal{X}$.

Then $$H_s = \begin{cases} \phi & \text{if} \quad s \leq 1 \\ A & \text{if} \quad 0 < s \leq 1 \\ A \bigcup A^c & \text{if} \quad s > 1 \end{cases}$$

$\Rightarrow \quad H_s = \{x \in \mathcal{X} / \xi_A(x) < s\}$ is semi-open for every $s \in \mathbb{R}$

$\Rightarrow \quad \xi_A \in \mathscr{SC}^-(\mathcal{X})$.

4. ALGEBRAIC PROPERTIES

4.1 Proposition: If $\lambda \geq 0$ and $f \in \mathscr{SC}^-(\mathcal{X})$ then $\lambda f \in \mathscr{SC}^-(\mathcal{X})$

Proof: If $\lambda = 0$ then $\lambda f \in \mathscr{SC}^-(\mathcal{X})$.

Now suppose that $\lambda > 0$. Let $\varepsilon > 0$ be given.

Let $x \in \mathcal{X}$ and U be a neighborhood of x_0 in $\mathcal{X}$.

Since f is $usqc$ at x_0, there exists a non-empty open set $W \subset U$ such that

$$f(x) - f(x_0) < \frac{\varepsilon}{\lambda} \quad \forall \quad x \in W$$

$\Rightarrow \quad (\lambda f)(x) - (\lambda f)(x_0) < \varepsilon \quad \forall \quad x \in W$

$\Rightarrow \quad \lambda f$ is $usqc$ at x_0. Hence $\lambda f \in \mathscr{SC}^-(\mathcal{X})$.

4.2 Proposition: If $\lambda \geq 0$ and $f \in \mathscr{SC}^+(\mathcal{X})$ then $\lambda f \in \mathscr{SC}^+(\mathcal{X})$

4.3 Proposition: If $f \in \mathscr{C}^-(\mathcal{X})$ and $g \in \mathscr{SC}^-(\mathcal{X})$ then $f + g \in \mathscr{SC}^-(\mathcal{X})$

Proof: Let $x \in \mathcal{X}$ and let $\varepsilon > 0$ be given.

Let U be a neighborhood of x_0 in $\mathcal{X}$.

Since f is usc at x_0, there exists a neighborhood G of x_0 in X such that

$$f(x) - f(x_0) < \frac{\varepsilon}{2} \quad \forall \quad x \in G.$$

Since g is $usqc$ at x_0, there exists a non-empty open set $W \subset U \bigcap G$ such that

$$g(x) - g(x_0) < \frac{\varepsilon}{2} \quad \forall \;\; x \in W.$$

$$\therefore \quad x \in W \quad \Rightarrow \quad (f+g)(x) - (f+g)(x_0) = f(x) - f(x_0) + g(x) - g(x_0)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for every $\varepsilon > 0$ and every neighborhood U of x_0, there exists a Non- empty open $W \subset U$ such that $(f+g)(x) - (f+g)(x_0) < \varepsilon \quad \forall \;\; x \in W$

$\Rightarrow \quad f+g$ is $usqc$ at x_0. Hence $f+g \in \mathscr{SC}^-(\mathcal{X})$.

4.4 Proposition: If $f \in \mathscr{C}^+(\mathcal{X})$ and $g \in \mathscr{SC}^+(\mathcal{X})$ then $f+g \in \mathscr{SC}^+(\mathcal{X})$.

4.5 Remark: If $f \in \mathscr{SC}^-(\mathcal{X})$ and $g \in \mathscr{SC}^-(\mathcal{X})$ then it is not necessary that $f+g \in \mathscr{SC}^-(\mathcal{X})$. Similarly $f \in \mathscr{SC}(\mathcal{X})$ and $g \in \mathscr{SC}^+(\mathcal{X})$ need not imply $f+g \in \mathscr{SC}^+(\mathcal{X})$. These facts are clear from the following example.

4.6 Example: Define $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 3 & \text{if} \quad 0 \le x < \frac{1}{2} \\ 1 & \text{if} \quad x = \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad x = \frac{1}{2} \\ 2 & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$$

then $(f+g)(x) = \begin{cases} 3 & \text{if} \quad 0 \le x < \frac{1}{2} \\ 1.5 & \text{if} \quad x = \frac{1}{2} \\ 2.5 & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$

Clearly f and g are $usqc$ at $x = \frac{1}{2}$ but $f+g$ is not.

Since f and g are $usqc$ on [0,1], $-f$ and $-g$ are $lsqc$ on [0,1]. But $-(f+g)$ is not $lsqc$ at $x=\frac{1}{2}$.

4.7 Proposition: If $f \in \mathscr{SC}^{+}(X)$, $g \in \mathscr{C}^{+}(X)$ and $f+g$ is continuous on X then f is semi-continuous on X.

Proof: Let $p \in X$ and $\varepsilon > 0$ be given. Let U be a neighborhood of p in X.

Since g is lsc at p, there exists a neighborhood G of p such that

$$-\frac{\varepsilon}{2} < g(x) - g(p) \quad \forall \; x \in G.$$

Since $f+g$ is continuous at p, there exists a neighborhood H of p such that

$$\left|(f+g)(x) - (f+g)(p)\right| < \frac{\varepsilon}{2} \quad \forall \; x \in H.$$

Since f is $lsqc$ at p, there exists a non-empty open set $W \subset U \cap G \cap H$ such that

$$-\varepsilon < f(x) - f(p) \quad \forall \; x \in W$$

$$\therefore \; x \in W \;\Rightarrow\; -\varepsilon < f(x) - f(p) < \frac{\varepsilon}{2} - g(x) + g(p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \; \left|(f(x) - f(p)\right| < \varepsilon$$

Thus for every $\varepsilon > 0$ and every neighborhood U of p in X, there exists a non- empty open set $W \subset U$ such that $\left|(f(x) - f(p)\right| < \varepsilon \quad \forall \; x \in W$

$\Rightarrow$ f is semi-continuous at p.

4.8 Proposition: If $f \in \mathscr{SC}^{-}(X)$, $g \in \mathscr{C}^{-}(X)$ and $f+g$ is continuous on X then f is semi-continuous on X.

4.9 Remark: From the following example it is clear that $f \wedge g$ and fg are not necessarily $lsqc$ ($usqc$) eventhough f and g are $lsqc$ ($usqc$).

4.10 Example: Define $f:[0,1] \to \mathbb{R}$ and $g:[0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad x = \frac{1}{2} \\ 0 & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad x = \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$$

Then $(f \wedge g)(x) = \begin{cases} 0 & \text{if} \quad x \in [0,1] - \left\{\frac{1}{2}\right\} \\ \frac{1}{2} & \text{if} \quad x = \frac{1}{2} \end{cases}$

It can be easily verified that f and g are $lsqc$ at $x = \frac{1}{2}$. But $f \wedge g$ and fg are not $lsqc$ at $x = \frac{1}{2}$.

4.11 Remark: Similar examples can be constructed to show $f \vee g$ and fg are not necessarily $usqc$ eventhough f and g are $usqc$.

4.12 Proposition:

(a) If $f \in \mathscr{SC}^+(\mathcal{X})$ and $g \in \mathscr{SC}^+(\mathcal{X})$ then $f \vee g \in \mathscr{SC}^+(\mathcal{X})$

(b) If $f \in \mathscr{SC}^-(\mathcal{X})$ and $g \in \mathscr{SC}^-(\mathcal{X})$ then $f \wedge g \in \mathscr{SC}^-(\mathcal{X})$

Proof: (a) Suppose that $f \in \mathscr{SC}^+(\mathcal{X})$ and $g \in \mathscr{SC}^+(\mathcal{X})$.

Then for every $s \in \mathbb{R}$ the sets $\{x \in \mathcal{X} / f(x) > s\}$ and $\{x \in \mathrm{X} / g(x) > s\}$ are semi-open. Since $\{x \in \mathcal{X} / (f \vee g)(x) > s\} = \{x \in \mathcal{X} / f(x) > s\} \bigcup \{x \in \mathcal{X} / g(x) > s\}$, the set $\{x \in \mathcal{X} / (f \vee g)(x) > s\}$ is semi-open for every $s \in \mathbb{R}$.

$\Rightarrow \quad f \vee g \in \mathscr{SC}^+(\mathcal{X})$.

The proof of (b) follows from (a) and by Proposition 2.6.

4.13 Proposition: Let $S \subset \mathscr{SC}^+(\mathcal{X})$. If $h(x) = \sup\{f(x) / f \in S\}$ exists for all $x \in \mathcal{X}$ then $h \in \mathscr{SC}^+(\mathcal{X})$.

Proof: Since $f \in \mathscr{SC}^+(\mathcal{X})$, the set $\{x \in \mathcal{X} / f(x) > s\}$ is semi-open for every $s \in \mathbb{R}$ Since $\{x \in \mathcal{X} / h(x) > s\} = \bigcup_{f \in S} \{x \in \mathcal{X} / f(x) > s\}$, the set $\{x \in \mathcal{X} / h(x) > s\}$ is semi-open for every $s \in \mathbb{R}$

$\Rightarrow \quad h \in \mathscr{SC}^+(\mathcal{X})$.

4.14 Proposition: Let $S \subset \mathscr{SC}^-(\mathcal{X})$. If $k(x) = \inf\{f(x) / f \in S\}$ exists for all $x \in \mathcal{X}$ then $k \in \mathscr{SC}^-(\mathcal{X})$

5. SOME OTHER CHARACTERIZATIONS

5.1 Definition: Let $f : \mathcal{X} \to \mathbb{R}$ be a bounded function.

We define $f^{\uparrow}(x) = \sup\{g(x) / g \in \mathscr{SC}^+(\mathcal{X}),\ g \le f\}$ $(x \in \mathcal{X})$

$$f^{\downarrow}(x) = \inf\{h(x) / h \in \mathscr{SC}^-(\mathcal{X}),\ h \ge f\}$$

5.2 Proposition: Let $f : \mathcal{X} \to \mathbb{R}$ and $g : \mathcal{X} \to \mathbb{R}$ be bounded functions. Then

(i) $f^{\uparrow} \le f \le f^{\downarrow}$

(ii) $f^{\uparrow} \in \mathscr{SC}^+(\mathcal{X})$ and $f^{\downarrow} \in \mathscr{SC}^-(\mathcal{X})$

(iii) If $h \le f, h \in \mathscr{SC}^+(\mathcal{X})$, then $h \le f^{\uparrow}$

(iv) If $k \ge f, k \in \mathscr{SC}^-(\mathcal{X})$, then $k \ge f^{\downarrow}$

(v) If $f \le g$ then $f^{\uparrow} \le g^{\uparrow}$ and $f^{\downarrow} \le g^{\downarrow}$

(vi) $\inf\{f^{\downarrow}(x) / x \in \mathcal{X}\} \ge \inf\{f(x) / x \in \mathcal{X}\} = \inf\{f^{\uparrow}(x) / x \in \mathcal{X}\}$

(vii) $\sup\{f^{\downarrow}(x) / x \in \mathcal{X}\} = \sup\{f(x) / x \in \mathcal{X}\} \ge \sup\{f^{\uparrow}(x) / x \in \mathcal{X}\}$

5.3 Proposition: Let $f : \mathcal{X} \to \mathbb{R}$ be a bounded function. Then

(i) $f \in \mathscr{SC}^{+}(\mathcal{X}) \iff f = f^{\uparrow}$

(ii) $f \in \mathscr{SC}^{-}(\mathcal{X}) \iff f = f^{\downarrow}$

(iii) f is semi-continuous on $\mathcal{X} \Rightarrow f^{\uparrow} = f^{\downarrow}$

6. UNIFORM LIMITS

6.1 Proposition: If $f_n \in \mathscr{SC}^{-}(\mathcal{X}),\ n = 1, 2, 3, \ldots,\ f_n \to f$ uniformly on $\mathcal{X}$ then $f \in \mathscr{SC}^{-}(\mathcal{X})$.

Proof: Let $\varepsilon > 0$ be given. Let $x_0 \in \mathcal{X}$ and U be a neighborhood of x_0 in $\mathcal{X}$.

Since $f_n \to f$ uniformly on $\mathcal{X}$, there exists an integer N such that $n \geq N$

$$\Rightarrow \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall \ x \in W$$

Since f_N is $usqc$ at x_0, there exists a non-empty open set $W \subset U$ such that

$$f_N(x) - f_N(x_0) < \frac{\varepsilon}{3} \quad \forall \ x \in W.$$

$$\begin{aligned} \therefore\ x \in W \ \Rightarrow\ f_N(x) - f_N(x_0) &= f(x) - f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Thus for every $\varepsilon > 0$ and every neighborhood U of x_0 there exists a non-empty open set $W \subset U$ such that

$$f(x) - f(x_0) < \varepsilon \quad \forall \ x \in W$$

$\Rightarrow \quad f$ is $usqc$ at x_0.

Hence $f \in \mathscr{SC}^{-}(\mathcal{X})$.

6.2 Proposition: If $f_n \in \mathscr{SC}^{+}(\mathcal{X}),\ n = 1, 2, 3, \ldots,\ f_n \to f$ uniformly on $\mathcal{X}$ then $f \in \mathscr{SC}^{+}(\mathcal{X})$.

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