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ON LEFT DERIVATIONS OF $d-$ALGEBRAS

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ABSTRACT

In this paper we investigate some properties of left derivations of $d-$algebras.

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1. INTRODUCTION

Y. Imai ([1], [2], [3]) and K.Isaki introduced two classes of abstract algebras: BCK algebras and BCI algebras. Q.P.Hu and X.Li introduced a broad class of abstract algebras: BCH algebras. ([4], [5]) J.Neggers and H.S.Kim introduced the notion of $d-$algebras. [6].

Y.B. Jun and X.L.X in [7] applied the notion of derivation in ring and near ring theory to BCI algebras and they also introduced a new concept called a regular derivation in BCI algebras. They investigated some of its properties, defined a $d-$invariant ideal and gave conditions for an ideal to be $d-$invariant. In non-commutative rings, the notion of derivations is extended to $d-$derivations, left derivations and central derivations.

In [8] J. Zhan and Y.L. Liu introduced the notion of $f-$derivations of BCI algebras. In particular they studied the regular $f-$derivations in detail and gave a characterization of regular $f-$derivations and characterized $p-$semi simple BCI algebras using the notion of regular $f-$derivation.

In [9] H.A. Abujabal and Nora O.Alshehri introduced the notion of left derivations of BCI algebras and investigated regular left derivations in BCI algebras. Recently, we have [10] introduced the notion of derivations on a $d-$algebra. In this paper we introduced the notion of left derivations on $d-$algebras and they investigated regular left derivations.

2. PRELIMINARIES

Definition 2.1: A $d-$algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:
1. $x * x = 0$
2. $0 * x = 0$
3. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$.

Definition 2.2: Let S be a non empty subset of a $d-$algebra X then, S is called $d-$sub algebra of X if $x * y \in S$ for all $x, y \in S$.

Definition2.3: Let X be a $d-$algebra and I be a subset of X then I is called $d-$ideal of X if it satisfies the following conditions:

1. $0 \in I$
2. $x * y \in I$ and $y \in I \Rightarrow x \in I$
3. $x \in I$ and $y \in X \Rightarrow x * y \in I$.

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Definition2.4: Let X be a $d-$algebra. A map $\theta : X \to X$ is a left –right derivation (briefly (l, r)-derivation) of X if it satisfies the identity $\theta(x * y) = (\theta(x) * y) \wedge (x * \theta(y))$ for all $x, y \in X$. If θ satisfies the identity $\theta(x * y) = (x * \theta(y)) \wedge (\theta(x) * y)$ for all $x, y \in X$, then θ is a right-left derivation (briefly (r, l)-derivation) of X. Moreover, if θ is both a (l, r)- and (r, l)-derivation, then θ is a derivation of X.

Definition 2.5: Let θ be a derivation of $d-$algebra$X$. An ideal $I$of X is said to be $\theta-$invariant if $\theta(I) \subseteq I$ where $\theta(I) = \{\theta(x) \mid x \in I\}$.

Definition 2.6: A self map θ of a $d-$algebra X is said to be regular if $\theta(0) = 0$.

Definition 2.7: Let $(X, *, 0)$ be a $d-$algebra and$x \in X$. Define $x * X$ = $\{x * a \mid a \in X\}$. X is said to be edge $d-$algebra if for any $x \in X, x * X = \{x, 0\}$.

Lemma 2.8: Let $(X, *, 0)$ be an edge $d-$algebra, then $x * 0 = x$ for any $x \in X$.

Lemma 2.9: If $(X, *, 0)$ is an $d-$algebra, then the condition $(x * (x * y)) * y = 0$ for all $x, y \in X$ holds.

Lemma 2.10: If $(X, *, 0)$ is an $d-$algebra, then $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

Lemma 2.11: Let $(X, *, 0)$ be an $d-$algebra then $y * (y * x) = x \quad \forall\ x, y \in X$.

3. LEFT DERIVATIONS

In this section we define the left derivations.

Definition 3.1: Let X be a $d-$algebra. By a left derivation of X we mean a self map θ of X satisfying

$$\theta(x * y) = (\theta(x) * y) \wedge (\theta(y) * x) \quad \forall\ x, y \in X.$$

Example 3.2: Let $X = \{0, 1, 2, 3\}$ be a $d-$algebra with Cayley table defined by

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	1	1	0	3
3	3	3	0	0

Define a map $\theta : X \to X$ by $\theta(x) = \begin{cases} 0 & if\ x = 0, 1, 3 \\ 3 & if\ x = 2 \end{cases}$

Then it is easily checked that θ is a left derivation of X.

Lemma 3.3: In any $d-$algebra X, the following properties hold for all $x, y, z \in X$.

1. $x * (x * (x * y)) = x * y$.
2. $x * 0 = 0 \quad \Rightarrow \quad x = 0$.
3. $((x * z) * (y * z)) * (x * y) = 0$.
4. $x \leq y \quad \Rightarrow \quad x * z \leq y * z$ and $z * y \leq z * x$.
5. $((x * y * (x * z)) * (z * y) = 0$.
6. $(x * z) * (y * z) = x * y$.
7. $(x * 0) * 0 = x$.
8. $x * a = x * b \quad \Rightarrow \quad a = b$.
9. $a * x = b * x \quad \Rightarrow \quad a = b$.
10. $x * y = 0 \quad \Rightarrow \quad x = y$.

Definition 3.4: A left derivation θ of a $d-$algebra X is said to be regular if $\theta(0) = 0$.

Lemma 3.5: Every left derivation of a $d-$algebra with $x*0=x$ is regular.

Proof: Now

$$\begin{aligned}\theta(0) &= \theta(0*x)\\ &= (\theta(0)*x)\wedge(\theta(x)*0)\\ &= (\theta(0)*x)\wedge\theta(x) \qquad (\because\ x*0=x)\\ &= \theta(x)*(\theta(x)*(\theta(0)*x))\\ \theta(0) &= \theta(0)*x.\end{aligned}$$

If $\theta(0)=0$, then nothing to prove. If $\theta(0)\neq 0$, then $\theta(0)*\theta(0)\neq 0*\theta(0)\neq 0$.

This is contradiction to the condition, $x*x=0$.

Hence $\theta(0)=0$. Therefore, every left derivation of a $d-$algebra with $x*0=x$ is regular.

Lemma 3.6: Let θ be a left derivation of a $d-$algebra X. Then for all $x,y\in X$ we have

1. $\theta(x)*x=\theta(y)*y$.
2. $\theta(x*y)=\theta(x)*y$.

Proof:
1. Let $x,y\in X$.

$$\begin{aligned}\theta(0) &= \theta(x*x)\\ &= (\theta(x)*x)\wedge(\theta(x)*x)\\ &= (\theta(x)*x)*((\theta(x)*x)*(\theta(x)*x))\\ &= (\theta(x)*x)*0\\ &= \theta(x)*x \qquad \cdots\cdots(1).\end{aligned}$$

Similarly, $\theta(0)=\theta(y)*y \quad \cdots\cdots(2)$.

From (1) and (2), $\theta(x)*x=\theta(y)*y$.

2. Let $x,y\in X$. Since θ be a left derivation of X.

$$\begin{aligned}\theta(x*y) &= (\theta(x)*y)\wedge(\theta(y)*x)\\ &= (\theta(y*x)*((\theta(y)*x)*(\theta(x)*y))\\ &= \theta(x)*y\end{aligned}$$

Lemma 3.7: Let θ be a left derivation of a $d-$algebra X such that $x*0=x$. Then $\theta(x)=x$ if and only if θ is regular.

Proof: Let θ be a regular.

That is $\theta(0)=0$.

$$\begin{aligned}Now\ \theta(0) &= \theta(x*x)\\ &= (\theta(x)*x)\wedge(\theta(x)*x)\\ &= (\theta(x)*x)*((\theta(x)*x)*(\theta(x)*x))\\ &= (\theta(x)*x)*0\\ &= \theta(x)*x\\ &= 0\end{aligned}$$

which implies $\theta(x)=x$.

Conversely, assume $\theta(x) = x$. Then it is clear that $\theta(0) = 0$. thus proving that θ is regular.

Theorem 3.8: Let θ be a left derivation of a $d-$algebra X. Then θ is regular if and only if every ideal of X is θ-invariant.

Proof: Let θ be a regular left derivation of a $d-$algebra X.

Then by lemma 3.7, $\theta(x) = x$ for all $x \in X$.

Let $y \in \theta(A)$, where A is an ideal of X.

Then $y = \theta(x)$ for some $x \in A$.

$$\begin{aligned} Thus\ y * x &= \theta(x) * x \\ &= x * x \\ &= 0 \in A \end{aligned}$$

Then $y \in A$ and $\theta(A) \subset A$.

Therefore A is $\theta-$invariant.

Conversely, let every ideal of X be $\theta-$invariant.

That is $\theta(A) \subset A$. Then $\theta(\{0\}) \subset \{0\}$. Hence $\theta(0) = 0$. Therefore θ is regular.

Theorem 3.9: Let X be a $d-$algebra. A self map θ of X is left derivation if and only if it is derivation.

Proof: Assume that θ is a left derivation of a $d-$algebra X.

$\theta(x * y) = \theta(x) * y = (x * \theta(y)) * ((x * \theta(y)) * (\theta(x) * y))$.

$\theta(x * y) = (\theta(x) * y) \wedge (x * \theta(y)) \qquad \cdots\cdots (1)$.

$$\begin{aligned} \theta(x * y) &= \theta(x) * y \\ &= (x * \theta(y)) \\ &= (\theta(x) * y) * ((\theta(x) * y) * (x * \theta(y)) \\ &= (x * \theta(y)) \wedge (\theta(x) * y) \qquad \cdots\cdots (2). \end{aligned}$$

From (1) and (2), θ is a derivation of X.

Conversely, let θ be a derivation of X. So it is a $(l, r)-$ derivation of X.

$$\begin{aligned} Now\ \theta(x * y) &= (\theta(x) * y) \wedge (x * \theta(y)) \\ &= (x * \theta(y)) * ((x * \theta(y)) * (\theta(x) * y)) \\ &= \theta(x) * y \\ &= (\theta(y) * x) * ((\theta(y) * x) * (\theta(x) * y)) \\ &= (\theta(x) * y) \wedge (\theta(y) * x). \end{aligned}$$

Hence θ is a left derivation of X.

Definition 3.10: Let X be a $d-$algebra and θ_1, θ_2 be two self maps of X. We have $\theta_1 \circ \theta_2 : X \to X$ as $(\theta_1 \circ \theta_2)(x) = \theta_1(\theta_2(x))\ \forall\ x \in X$.

Lemma 3.11: Let $(X, *, 0)$ be a $d-$algebra. Let θ_1 and θ_2 be two left derivations of X, then $\theta_1 \circ \theta_2$ is also a left derivation of X.

Proof:

$$\begin{aligned}(\theta_1 \circ \theta_2)(x*y) &= \theta_1(\theta_2(x*y)) \\ &= \theta_1((\theta_2(x)*y) \wedge (\theta_2(y)*x)) \\ &= \theta_1[(\theta_2(y)*x)*[(\theta_2(y)*x)*\theta_2(x)*y)] \\ &= (\theta_1(\theta_2(x))*y) \wedge \theta_1(\theta_2(y))*x \\ &= ((\theta_1 \circ \theta_2)(x)*y) \wedge ((\theta_1 \circ \theta_2)(y)*x)\end{aligned}$$

Hence $\theta_1 \circ \theta_2$ is a left derivation of X.

It can be easily proved that

Theorem 3.12: Let $(X,*,0)$ be a $d-$algebra and θ_1, θ_2 are left derivations of X. Then $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$.

Definition 3.13: Let X be a $d-$algebra and θ_1, θ_2 be two self maps of X. We define $\theta_1 \cdot \theta_2 : X \to X$ as $(\theta_1 \cdot \theta_2)(x) = \theta_1(x) \cdot \theta_2(x) \;\; \forall\, x \in X.$

Theorem 3.14: Let $(X,*,0)$ be a $d-$algebra and θ_1, θ_2 are left derivations of X. Then $\theta_1 \cdot \theta_2 = \theta_2 \cdot \theta_1$.

Proof: Let X be a $d-$algebra and θ_1, θ_2 are left derivations of X.

$$\begin{aligned}Now\ (\theta_1 \cdot \theta_2)(x*y) &= \theta_1(x*y)\cdot\theta_2(x*y) \\ &= [(\theta_1(x)*y) \wedge (\theta_1(y)*x)]\cdot\theta_2(x*y) \\ &= 0 \quad on\ simplification \quad \cdots\cdots(1). \\ Similarly\ (\theta_2 \cdot \theta_1)(x*y) &= \theta_2(x*y)\cdot\theta_1(x*y) \\ &= 0 \qquad\qquad \cdots\cdots(2).\end{aligned}$$

From (1) and (2), $(\theta_1 \cdot \theta_2)(x*y) = (\theta_2 \cdot \theta_1)(x*y)$.

Putting $y = 0$ we get for all $x \in X$,

$(\theta_1 \cdot \theta_2)(x) = (\theta_2 \cdot \theta_1)(x)$. Hence $\theta_1 \cdot \theta_2 = \theta_2 \cdot \theta_1$

Notation: Der(X) denote the set of all left derivations on X.

Definition 3.15: Let $\theta_1, \theta_2 \in$ Der(X). Define the binary operation $\wedge$ as

$$(\theta_1 \wedge \theta_2)(x) = \theta_1(x) \wedge \theta_2(x).$$

It is easy to prove that

Lemma 3.16: Let X be a $d-$algebra and θ_1, θ_2 are left derivations of X. Then $\theta_1 \wedge \theta_2$ is also a left derivation of X.

Lemma 3.17: Let X be a $d-$algebra. If $\theta_1, \theta_2, \theta_3 \in$ Der(X). Then

$$\theta_1 \wedge (\theta_2 \wedge \theta_3) = (\theta_1 \wedge \theta_2) \wedge \theta_3.$$

Proof: Let X be a $d-$algebra and $\theta_1, \theta_2, \theta_3$ are left derivations of X.

$$\begin{aligned}Now\ ((\theta_1 \wedge \theta_2) \wedge \theta_3)(x*y) &= (\theta_1 \wedge \theta_2)(x*y) \wedge \theta_3(x*y) \\ &= \theta_3(x*y)*(\theta_3(x*y)*(\theta_1 \wedge \theta_2)(x*y)) \\ &= (\theta_1 \wedge \theta_2)(x*y) \\ &= (\theta_2(x)*y)*((\theta_2(x)*y)*(\theta_1(x)*y)) \\ &= \theta_1(x)*y \qquad\qquad \cdots\cdots(1).\end{aligned}$$

Also consider the following

$$\begin{aligned}\theta_1 \wedge (\theta_2 \wedge \theta_3)(x*y) &= \theta_1(x*y) \wedge (\theta_2 \wedge \theta_3)(x*y)\\ &= \theta_1(x*y) \wedge [\theta_2(x)*y) \wedge \theta_3(x*y)]\\ &= \theta_1(x*y) \wedge [\theta_3(x*y) * ((\theta_3(x*y)) * (\theta_2(x*y)))]\\ &= \theta_1(x)*y \qquad \cdots\cdots(2).\end{aligned}$$

This implies that $(\theta_1 \wedge (\theta_2 \wedge \theta_3))(x*y) = ((\theta_1 \wedge \theta_2) \wedge \theta_3)(x*y)$.

Put $y = 0$, we have

$(\theta_1 \wedge (\theta_2 \wedge \theta_3))(x) = ((\theta_1 \wedge \theta_2) \wedge \theta_3)(x)$.
$\Rightarrow \ \theta_1 \wedge (\theta_2 \wedge \theta_3) = (\theta_1 \wedge \theta_2) \wedge \theta_3$.

From the above two lemmas we obtain the following.

Theorem 3.18 (Der (X), $\wedge$) is a semi group.

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