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A COMMON FIXED POINT THEOREM FOR SELF MAPS ON A PROBABILISTIC METRIC SPACE UNDER DNR COMMUTATIVITY ONDITION

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ABSTRACT

The aim of present paper is to obtain a common fixed point theorem for two maps and hence for a sequence of mappings with respect to another two self maps on a probabilistic metric space through DNR-commutativity property, the property (E.A) and implicit relations.

These results generalize the result of Mukesh Sharma and Dimri [9].

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1. INTRODUCTION AND PRELIMINARIES

In 1942, K. Menger [7] introduced the notion of probabilistic metric space (briefly PM-space) as a generalization of metric space. The development of fixed point theory in PM- spaces was due to Schweizer and Sklar [11, 12]. Sehgal [13] initiated study of contraction mapping theorems in PM-spaces. Ciric and Milovanovic - Arandjelovic [2] introduced the notion of pointwise R-weakly commutativity to PM-spaces. Pant [10] introduced the notion of reciprocal continuity and obtained common fixed point theorems in metric spaces using R-weak commutativity and reciprocal continuity of mappings, Kumar and Chugh [4] established common fixed point theorems in metric spaces.

Mihet [8] established a fixed point theorem concerning probabilistic contractions satisfying an implicit relation. S. Kumar and B.D. Pant [5] established common fixed point theorems in PM- spaces using implicit relations. J.K. Kohli, S. Vasista and D. Kumar [3] extended the result of [5] to six mappings.

Recently Aamri and Moutanakil [1] and Liu, J. wu and Z. Li [6] defined the property (E.A) and the common property (E.A) respectively and established some results by using the properties in metric spaces.

Mukesh Sharma and Dimri [9] established a common fixed point theorem for a sequence of self mappings on a probabilistic metric space satisfying pointwise R-weakly commutativity and property (E.A) and using an implicit relation.

In this paper, we introduce the notion of DNR-commutativity in PM-spaces, which includes the notion of pointwise R-weak commutativity. Using this new notion and property (E.A), under certain implicit relation, we establish a common fixed point theorem for a pair of self maps with respect to another pair of self maps on a probabilistic metric space and extend it to a sequence of self maps which in turn includes the result of Mukesh Sharma and Dimri [9].

Throughout the paper, $\mathbb{R}$ stands for the real line and $\mathbb{R}^+$ stands for the set of non negative real numbers. We begin with some definitions.

Definition 1.1: [12] A mapping $F: \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

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We shall denote by $\mathfrak{D}$, the class of all distribution functions.

Definition 1.2: [12] A probabilistic metric space is a pair (X, F) where X is a non-empty set and F is a mapping from $X \times X \to \mathfrak{D}$. For $(u, v) \in X \times X$, the distribution function $F(u, v)$ is denoted by $F_{u,v}$. The functions $F_{u,v}$ are assumed to satisfy the following conditions.

(P$_1$) $F_{u,v}(x) = 1$ for all $x > 0$ if and only if $u = v$,
(P$_2$) $F_{u,v}(0) = 0$ for all $u, v \in X$,
(P$_3$) $F_{u,v}(x) = F_{v,u}(x)$ for every $u, v \in X$,
(P$_4$) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$ for all $u, v, w \in X$ and $x, y > 0$.

Definition 1.3: [12] A mapping $\Delta: [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (briefly t -norm) if the following conditions are satisfied.
(i) $\Delta(a, 1) = a \quad \forall a \in [0,1]$
(ii) $\Delta(a, b) = \Delta(b, a) \quad \forall a, b \epsilon [0,1]$
(iii) If $c \geq a$ *and* $d \geq b$ *then* $\Delta(c, d) \geq \Delta(a, b) \forall a, b, c, d \epsilon [0,1]$
(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c)) \quad \forall a, b, c \in [0,1]$

Example 1.4: (i) $\Delta(a, b) = min\{a, b\}$
(ii) $\Delta(a, b) = ab$ and (iii) $\Delta(a, b) = min\{a + b - 1, 0\}$ are some t-norms.

Definition 1.5: [12] A Manger PM-spaceis a triplet(X, F, Δ), where (X, F) is a PM-space and t is a t-norm with the following condition:

$$F_{u,v}(x + y) \geq \Delta\left(F_{u,w}(x), F_{w,v}(y)\right) \ \forall \ x, y \geq 0 \ and \ u, v, w \in X.$$

Definition 1.6: [2] Two self mappings A and S of a PM-space (X, F) are said to be pointwise R-weakly commuting if given $z \in X$, there exists$R_z > 0$ such that

$$F_{ASz,SAz}(t) \geq F_{Az,Sz}\left(\frac{t}{R_z}\right) \text{ for } t > 0.$$

Definition 1.7: [1] A pair (A, S) of self mappings of a PM space (X, F) is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 1.8: [6] Two pairs (A, S) and (B, T) of self mappings of a PM-space (X, F) are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}, \{y_n\} \in X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$ for some $z \in X$.

2. IMPLICIT RELATION

Definition 2.1: [9] Let Φ be the class of all real valued continuous functions $\varphi: (\mathbb{R}^+)^4 \to \mathbb{R}$, non deceasing in first argument and satisfying the following conditions:

for all $x, y \geq 0$, $\quad \varphi(x, y, x, y) \geq 0$ (or) $\varphi(x, y, y, x) \geq 0 \Rightarrow x \geq y$ (2.1.1)

$$\varphi(x, x, 1,1) \geq 0 \ for \ all \ x \geq 1 \quad (2.1.2)$$

Members of Φ are called implicit relations.

Definition 2.2: Let X be a non empty set,Ψ denote the class of all functions $\psi: X \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\psi(x, t) > 0 \ for \ all \ x \in X \ and \ t > 0$.

Members of Ψ are called DNR functions with respect to X.

Definition 2.3: Two self mappings A and S of a PM-space (X, F) are said to be DNR-commutating if there exists $\psi \in \Psi$ such that

$F_{ASz,SAz}(t) \geq F_{Az,Sz}(\psi(z, t))$ for all $z \in X$ and $t > 0$.

We observe that if A and S are point wise R- weakly commuting self maps on a PM- space X, then A and S are DNR-commuting.

Mukesh Sharma and Dimri [9] proved the following lemma and theorem.

Lemma 2.4: [9] Let $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}, S$ and T be self maps of a Menger space (X, F, Δ) satisfying the following conditions

$$A_i(X) \subseteq T(X),\ A_o(X) \subseteq S(X) \tag{2.4.1}$$

There exists $\varphi \in \Phi$ and $h \in (0,1)$ such that

$$\varphi\left(F_{A_ix,A_0y}(ht), F_{Sx,Ty}(t), F_{A_ix,Sx}(t), F_{A_oy,Ty}(ht)\right) \geq 0 \tag{2.4.2}$$

for all $x, y \in X, t > 0$.

Suppose that (A_0, T) satisfies property (E.A). Then the pairs (A_i, S) and (A_0, T) have the common property (E.A).

Theorem 2.5: [9] Let $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}, S$ and T be self maps of a Menger space (X, F, Δ) satisfying the conditions (2.4.1) and (2.4.2) of Lemma 2.4, (A_0, T) satisfies the property (E.A) and the pairs (A_i, S) and (A_0, T) are point wise R-weakly commuting. If range of one of S and T is a closed subspace of X, then $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}, S$ and T have a unique common fixed point.

3. MAIN RESULTS

We prove our main theorem by using DNR commuting property instead of point wise R-weakly commuting property and our theorem is a generalization of Theorem 2.5. For this first we prove our theorem to four self maps and later extend to a sequence of self maps.

We also provide an example of a pair of maps which are DNR-commuting.

Theorem 3.1: Let A_0, A_1, S and T be self maps of a PM-space satisfying the conditions (2.4.1) and (2.4.2) of Lemma 2.4, (A_0, T) satisfies the property (E.A) and the pairs (A_1, S) and (A_0, T) are DNR- commuting. If one of $S(X)$ and $T(X)$ is a closed subspace of X, then A_0, A_1, S and T have a unique common fixed point.

Proof: In view of Lemma 2.4 the pairs (A_1, S) and (A_0, T) have the common property (E.A).

Hence there exist sequences $\{x_n\}$ and $\{y_n\}$ $in\ X$ such that

$\lim_{n\to\infty} A_0x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} A_1y_n = \lim_{n\to\infty} Sy_n = z$ for some $z \in X$.

Suppose $S(X)$ is a closed subspace of X. Then there exists $u \in X$ such that $Su = z$.

Now we claim that $A_1u = z$.

Putting $x = u\ and\ y = x_n$ in (2.4.2), we get

$$\varphi\left(F_{A_1u,A_0x_n}(ht), F_{Su,Tx_n}(t), F_{A_1u,Su}(t), F_{A_ox_n,Tx_n}(ht)\right) \geq 0$$

On letting $n \to \infty$, we have

$$\varphi\left(F_{A_1u,z}(ht), F_{z,z}(t), F_{A_1u,z}(t), F_{z,z}(ht)\right) \geq 0$$

i.e. $\varphi\left(F_{A_1u,z}(ht), 1, F_{A_1u,z}(t), 1\right) \geq 0$

Since φ is non decreasing, (2.1.1) gives $F_{A_1u,z}(ht) \geq 1$

Hence $A_1u = z$.

Thus we have $z = Su = A_1u$.

Since $A_1(X) \subseteq T(X)$, there exists $v \in X$ such that $z = A_1u = Tv$.

We claim that $A_0 v = z$.

Putting $x = y_n \; and \; y = v$ in (2.4.2), we get

$\left(F_{A_1 y_n, A_0 v}(ht), F_{Sy_n, Tv}(t), F_{A_1 y_n, Sy_n}(t), F_{A_o v, Tv}(ht)\right) \geq 0$

On letting $n \to \infty$, we have

$\left(F_{z, A_0 v}(ht), F_{z,z}(t), F_{z,z}(t), F_{A_o v, z}(ht)\right) \geq 0$

i.e. $\left(F_{z, A_0 v}(ht), 1, 1, F_{A_o v, z}(ht)\right) \geq 0$.

Therefore (2.1.1) gives that $F_{A_o v, z}(ht) \geq 1$.

Hence $A_o v = z$.

Thus we have $z = Su = A_1 u = Tv = A_0 v$.

Since A_1, S are DNR-commuting, there exists $\psi \in \Psi$ such that

$F_{A_1 Su, SA_1 u}(t) \geq F_{A_1 u, Su}\left(\psi(u,t)\right) = 1$

i.e. $A_1 Su = SA_1 u$ and hence $A_1 Su = SA_1 u = A_1 A_1 u = SSu$.

Also A_0 and T are DNR-commuting. Hence there exists $\psi \in \Psi$ such that

$F_{A_0 Tv, TA_0 v}(t) \geq F_{A_0 v, Tv}\left(\psi(v,t)\right) = 1$

i.e. $A_0 Tv = TA_0 v$ and $A_0 Tv = TA_0 v = A_0 A_0 v = TTv$.

Now putting $x = A_1 u \; and \; y = v$ in (2.4.2), we get

$\varphi\left(F_{A_1 A_1 u, A_0 v}(ht), F_{SA_1 u, Tv}(t), F_{A_1 A_1 u, SA_1 u}(t), F_{A_o v, Tv}(ht)\right) \geq 0$

i.e. $\varphi\left(F_{A_1 A_1 u, A_1 u}(ht), F_{A_1 A_1 u, A_1 u}(t), 1, 1\right) \geq 0$.

Since φ is non decreasing (2.1.2) gives $F_{A_1 A_1 u, A_1 u}(t) \geq 1$

i.e. $A_1 A_1 u = A_1 u \Rightarrow A_1 z = z$ and $A_1 z = z = Sz$.

Now putting $x = u \; and \; y = A_0 v$ in (2.4.2), we get

$\varphi\left(F_{A_1 u, A_0 A_0 v}(ht), F_{Su, TA_0 v}(t), F_{A_1 u, Su}(t), F_{A_o A_0 v, TA_0 v}(ht)\right) \geq 0$

i.e. $\varphi\left(F_{A_0 v, A_0 A_0 v}(ht), F_{A_0 v, A_0 A_0 v}(t), 1, 1\right) \geq 0$

i.e. $A_0 v = A_0 A_0 v$ (using (2.1.2), since φ is non deceasing)

$\therefore z = A_0 z$ and $z = A_0 z = Tz$

which gives $z = A_1 z = Sz = A_0 z = Tz$.

Hence z is a common fixed point for $A_0, A_1, S \; and \; T$.

Let if possible p be another fixed point of $A_0, A_1, S \; and \; T$.

Then $A_0 p = A_1 p = Sp = Tp = p$.

Now putting $x = z\ and\ y = p$ in (2.4.2), we get

$\varphi\big(F_{A_1 z, A_0 p}(ht), F_{Sz,Tp}(t), F_{A_1 z, Sz}(t), F_{A_o p, Tp}(ht)\big) \geq 0$

i.e. $\varphi\big(F_{z,p}(ht), F_{z,p}(t), F_{z,z}(t), F_{p,p}(ht)\big) \geq 0$

i.e. $\varphi\big(F_{z,p}(ht), F_{z,p}(t), 1,1\big) \geq 0$

i.e. $F_{z,p}(t) \geq 1$ ($\because$ by (2.1.2) and φ is non deceasing)

$\therefore z = p$

Hence z is the unique common fixed point of $A_0, A_1, S\ and\ T$.

Now, we prove a common fixed point theorem for a sequence of self maps which are DNR commuting in pairs.

Theorem 3.2: Let $\{A_i\}_{i \in \mathbb{N} \cup \{0\}}, S$ and T be self maps of a PM space (X, F) satisfying the conditions (2.4.1) and (2.4.2) of Lemma 2.4, (A_0, T) satisfies the property (E.A) and the pairs (A_i, S) and (A_0, T) are DNR commuting. If range of one of S and T is a closed subspace of X, then $\{A_i\}_{i \in \mathbb{N} \cup \{0\}}, S$ and T have a unique common fixed point.

Proof: Let $z_i . i > 1$ be the common fixed point of $A_0, A_i, S\ and\ T$.

In (2.4.2), put $x = z_2, y = z_2$ and $i = 1$, we get

$\varphi\big(F_{A_1 z_2, A_0 z_2}(ht), F_{Sz_2, Tz_2}(t), F_{A_1 z_2, Sz_2}(t), F_{A_o z_2, Tz_2}(ht)\big) \geq 0$

$\Rightarrow \varphi\big(F_{A_1 z_2, z_2}(ht), F_{z_2, z_2}(t), F_{A_1 z_2, z_2}(t), F_{z_2, z_2}(ht)\big) \geq 0$

$\Rightarrow \varphi\big(F_{A_1 z_2, z_2}(ht), 1, F_{A_1 z_2, z_2}(t), 1\big) \geq 0$

$\Rightarrow A_1 z_2 = z_2$ ($\because \varphi$ is non decreasing, by (2.1.2))

$\therefore z_2$ is fixed point of A_1.

Thus z_2 is a fixed point of $A_0, A_1, S\ and\ T$, so that $z_1 = z_2$, by uniqueness of common fixed point.

In a similar manner, putting $x = z_i, y = z_i$ and $A_i = A_1$ in (2.4.2), we get $A_1 z_i = z_i$ and hence $z_i = z_1\ for\ all\ i > 1$.

Thus z_1 is a common fixed point of $A_0, A_1, A_2, \ldots, A_i, \ldots\ , S\ and\ T$.

Note: Theorem 2.5 is a simple corollary of Theorem 3.2.

Now, we give an example to illustrate DNR commuting mappings.

Example 3.3: Let $X = \{2,3,4,\ldots\}$ with the metric $d(x, y) = |x - y|$ and define

$$F_{x,y}(t) = \begin{cases} 0 & if\ \ t \leq x \\ 1 & if\ \ t > y \\ \dfrac{t - x}{y - x} & if\ \ x < t \leq y \end{cases}$$

for $x < y$.

Clearly (X, F) is a PM-space.

Define $\psi(x, t) = \begin{cases} x & if\ \ t \leq x \\ \frac{t-1}{x} & if\ \ t > x \end{cases}$ for $x \in [2, \infty)$

Then ψ is a DNR function.

Define $A, S: X \to X$ by $Ax = x + 1, Sx = x^2$.

Then for $z \in X$, $ASz = z^2 + 1$ and $SAz = (z+1)^2$.

Clearly $z^2 + 1 < (z+1)^2$ for $z \in X$.

Claim $F_{z^2+1,(z+1)^2}(t) \geq F_{z+1,z^2}(\psi(z,t))$ for all $t > 0$ (3.3.1)

Case I: $t \leq z^2 + 1$

Then L.H.S of (1) is 0 and

$$t \leq z \Rightarrow \psi(z,t) = z \Rightarrow F_{z+1,z^2}\big(\psi(z,t)\big) = 0$$
$$t > z \Rightarrow \psi(z,t) = \frac{t-1}{z} \leq z \Rightarrow F_{z+1,z^2}\big(\psi(z,t)\big) = 0$$

Case II: $t \geq (z+1)^2$

Then L.H.S of (3.3.1) is $1 \geq F_{z+1,z^2}\big(\psi(z,t)\big) = 0$

Case III: $z^2 + 1 < t < (z+1)^2$ (3.3.2)

L.H.S of (3.3.1) = $\frac{t-(z^2+1)}{(z+1)^2-(z^2+1)} = \frac{t-(z^2+1)}{2z}$

From (3.3.2), $z < z^2 + 1 < t \Rightarrow \psi(z,t) = \frac{t-1}{z}$ and
$z < \frac{t-1}{z} = \psi(z,t) < z + 2$ ($\because z^2 + 1 < t < (z+1)^2$)

If $z + 1 \geq \frac{t-1}{z} = \psi(z,t)$, then R.H.S of (3.3.1) is '0'.

Suppose $z + 1 < \frac{t-1}{z} < z + 2 \leq z^2$

Then R.H.S of (3.3.1) = $\frac{\frac{t-1}{z}-(z+1)}{z^2-(z+1)} = \frac{t-(z^2+z+1)}{z(z^2-(z+1))}$

Claim: $\frac{t-(z^2+1)}{2z} \geq \frac{t-(z^2+z+1)}{z(z^2-(z+1))}$ (3.3.3)

i.e. $\frac{t-(z^2+1)}{2} \geq \frac{t-(z^2+z+1)}{(z^2-(z+1))}$

For $z = 2$, (3.3.3) holds since $t < (z+1)^2 = 9$

Now for $z \geq 3$

We have $2 \leq z^2 - (z+1)$ so that

$$\frac{t-(z^2+1)}{2} \geq \frac{t-(z^2+1)}{z^2-(z+1)} \geq \frac{t-(z^2+z+1)}{z^2-(z+1)}$$

Hence (3.3.3) holds

Thus $F_{ASz,SAz}(t) \geq F_{Az,Sz}(\psi(z,t))$.

Hence the pair (A,S) is DNR commuting.

Note: The maps A ans S of the above example do not have a common fixed point and do not have property (E.A). Thus Example 3.3 shows that in the absence of property (E.A), DNR commutativity alone may not guarantee the existence of a common fixed point.

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