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ON CERTAIN SUMMATION FORMULAE BASED ON HALF ARGUMENT ASSOCIATED TO HYPERGEOMETRIC FUNCTION

SALAH UDDIN

P.D.M College of Engineering, Bahadurgarh , Haryana , India
E-mails : sludn@yahoo.com ; vsludn@gmail.com

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ABSTRACT

The main objective of this paper is to establish certain summation formulae based on half argument involving Gauss second summation theorem. The results derived in this paper are of general character.

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A. INTRODUCTION:

The Pochhammer's symbol is defined by

$$(\alpha\ ,\ k\ ) = (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)(\alpha+2)\ldots.(\alpha+k-1); & \text{if } k = 1,2,3\ldots\ldots \\ 1 & ;\quad \text{if } k=0 \\ k! & ;\quad \text{if } \alpha = 1 \end{cases} \tag{1}$$

Generalized Gaussian Hypergeometric function of one variable is defined by

$${}_AF_B(a_1,a_2,\ldots,a_A;b_1,b_2,\ldots b_B;z\ ) = \sum_{k=0}^{\infty} \frac{(a_1)k(a_2)k\ldots..(a_A)k\ z^k}{(b_1)k(b_2)k\ldots..(b_B)k\ k!}$$

or

$${}_AF_B((a_A);(b_B);z) \equiv {}_AF_B((a_j)^A{}_{j=1}\ ;(bj)^B{}_{j=1}\ ;\ z) = \sum_{k=0}^{\infty} \frac{((a_A))k\ z^k}{((b_B))k\ k!} \tag{2}$$

where the parameters b_1 , b_2 , ….,b_B are neither zero nor negative integers and A , B are non negative integers.

Contiguous Relation is defined by
[Andrews's p.363 (9.16), E.D. p.51 (10), H.T.F.I. p.103 (32)]

$$(a-b)\ {}_2F_1(a,\ b\ ;\ c;\ z) = a\ {}_2F_1(a+1,b;\ c;\ z) - b\ {}_2F_1(a,b+1;\ c;\ z) \tag{3}$$

Legendre's duplication formula:

$$\surd(\pi)\Gamma(2z) = 2^{(2z-1)}\ \Gamma(z)\ \Gamma(z+\tfrac{1}{2}) \tag{4}$$

$$\Gamma(\tfrac{1}{2}) = \surd(\pi) = 2^{(b-1)}\ \Gamma(\tfrac{b}{2})\ \Gamma(\tfrac{b+1}{2})\ /\Gamma(b) \tag{5}$$

$$= 2^{(a-1)}\ \Gamma\ (\tfrac{a}{2})\ \Gamma(\tfrac{a+1}{2})$$
(6)

Recurrence relation is defined by

$$\Gamma\ (z+1) = z\ \Gamma\ (z) \tag{7}$$

Corresponding author: SALAHUDDIN, E-mail: sludn@yahoo.com

Gauss second summation theorem is defined by [Prud. 491(7.3, 7.4)]

$$ {}_2F_1(a, b; c; \tfrac{1}{2}) = \Gamma(\tfrac{a+b+1}{2})\,\Gamma(\tfrac{1}{2}) \,/\, \Gamma(\tfrac{a+1}{2})\,\Gamma(\tfrac{b+1}{2}) \tag{8} $$

$$ = 2^{(b-1)}\,\Gamma(\tfrac{b}{2})\,\Gamma(\tfrac{a+b+1}{2}) \,/\, \Gamma(b)\,\Gamma(\tfrac{a+1}{2}) \tag{9} $$

In a monograph of Prudnikov et al. , a summation theorem is given in the form [Prud. , p.491(7.3,7.5)]

$$ {}_2F_1\,[a ,b; \tfrac{a+b-1}{2} ; \tfrac{1}{2}] = \sqrt{(\pi)}[\,\Gamma(\tfrac{a+b+1}{2}) \,/\, \Gamma(\tfrac{a+1}{2})\,\Gamma(\tfrac{b+1}{2}) + 2\,\Gamma(\tfrac{a+b-1}{2}) \,/\, \Gamma(a)\,\Gamma(b)\,] \tag{10} $$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$$ {}_2F_1\,[a , b; \tfrac{a+b-1}{2} ; \tfrac{1}{2}] = 2^{(b-1)}\,\Gamma(\tfrac{a+b-1}{2}) \,/\, \Gamma(b)[\,\Gamma(\tfrac{b}{2}) \,/\, \Gamma(\tfrac{a-1}{2}) + 2^{(a-b+1)}\,\Gamma(\tfrac{a}{2})\,\Gamma(\tfrac{a+1}{2})/\{\Gamma(a)\}^2 + \Gamma(\tfrac{b+2}{2}) \,/\, \Gamma(\tfrac{a+1}{2})] \tag{11} $$

B MAIN RESULTS OF SUMMATION FORMULAE:

$$ {}_2F_1(a, b ; \tfrac{a+b+9}{2}; \tfrac{1}{2}) = 2^b\,\Gamma(\tfrac{a+b+9}{2})/(a-b)\,\Gamma(b)\,[\Gamma(\tfrac{b}{2}) \,/\, \Gamma(\tfrac{a+1}{2}) $$

$$ \{\frac{8a(-15+23a-9a^2+a^3+49b-14ab+21a^2b+7b^2+35ab^2+7b^3)}{(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)(a-b+7)} + \frac{8b(-15+49a+7a^2+7a^3+23b-14ab+35a^2b-9b^2+21ab^2+b^3)}{(a-b-7)(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)}\} $$

$$ -\Gamma(\tfrac{b+1}{2}) \,/\, \Gamma(\tfrac{a}{2})\{\frac{16(15+49a-7a^2+7a^3+23b+14ab+35a^2b+9b^2+21ab^2+b^3)}{(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)(a-b+7)} + \frac{16(15+23a+9a^2+a^3+49b+14ab+21a^2b-7b^2+35ab^2+7b^3)}{(a-b-7)(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)}\}] \tag{12} $$

$$ {}_2F_1(a, b ; \tfrac{a+b+10}{2}; \tfrac{1}{2}) = 2^b\,\Gamma(\tfrac{a+b+10}{2})/(a-b)\,\Gamma(b)\,[\Gamma(\tfrac{b}{2}) \,/\, \Gamma(\tfrac{a}{2}) $$

$$ \{\frac{16(-48a+44a^2-12a^3+a^4+48b+168ab-28a^2b+28a^3b+44b^2+28ab^2+70a^2b^2+12b^3+28ab^3+b^4)}{(a-b-6)(a-b-4)(a-b-2)(a-b)(a-b+2)(a-b+4)(a-b+6)(a-b+8)} $$

$$ + \frac{128b(16a+2a^2+a^3+16b+7a^2b-2b^2+7ab^2+b^3)}{(a-b-8)(a-b-6)(a-b-4)(a-b-2)(a-b)(a-b+2)(a-b+4)(a-b+6)} - (\tfrac{b+1}{2}) \,/\Gamma(\tfrac{a+1}{2})\{ \frac{128a(16a-2a^2+a^3+16b+7a^2b+2b^2+7ab^2+b^3)}{(a-b-6)(a-b-4)(a-b-2)(a-b)(a-b+2)(a-b+4)(a-b+6)(a-b+8)} $$

$$ + \frac{16(48a+44a^2+12a^3+a^4-48b+168ab+28a^2b+28a^3b+44b^2-28ab^2+70a^2b^2-12b^3+28ab^3+b^4)}{(a-b-8)(a-b-6)(a-b-4)(a-b-2)(a-b)(a-b+2)(a-b+4)(a-b+6)}\}] \tag{13} $$

$$ {}_2F_1(a, b ; \tfrac{a+b+11}{2}; \tfrac{1}{2}) = 2^b\Gamma(\tfrac{a+b+11}{2})/(a-b)\Gamma(b)[\,\Gamma(\tfrac{b}{2}) \,/\Gamma(\tfrac{a+1}{2})\,\{ $$

$$ \frac{16a(105-176a+86a^2-16a^3+a^4-24b+492ab-72a^2b+36a^3b+270b^2+126a^2b^2+24b^3+84ab^3+9b^4)}{(a-b-7)(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)(a-b+7)(a-b+9)} $$

$$ +\frac{16b(105-24a+270a^2+24a^3+9a^4-176b+492ab+84a^3b+86b^2-72ab^2+126a^2b^2-16b^3+36ab^3+b^4)}{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)(a-b+7)} $$

$$ [\Gamma(\tfrac{b}{2}) \,/\Gamma(\tfrac{b+1}{2})\,\{ \frac{32(105+24a+270a^2-24a^3+9a^4+176b+492ab+84a^3b+86b^2+72ab^2+126a^2b^2+16b^3+36ab^3+b^4)}{(a-b-7)(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)(a-b+7)(a-b+9)} + $$

$$ \frac{32(105+176a+86a^2+16a^3+a^4+24b+492ab+72a^2b+36a^3b+270b^2+126a^2b^2-24b^3+84ab^3+9b^4)}{(a-b-9)(a-b-7)(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)(a-b+7)}\}] \tag{14} $$

C DERIVATIONS OF SUMMATION FORMULAE (12) TO (14):

Derivation of (12): Substituting $c = \frac{a+b+9}{2}$ and $z = \frac{1}{2}$ in equation (2) , we get

$$ (a-b)\,{}_2F_1(a, b; \tfrac{a+b+9}{2}; \tfrac{1}{2}) = a\,{}_2F_1(a+1 , b ; \tfrac{a+b+9}{2} ; \tfrac{1}{2}) - b\,{}_2F_1(a , b+1 ; \tfrac{a+b+9}{2} ; \tfrac{1}{2}) $$

Now with the help of Gauss second summation theorem , we get

$$ \text{L.H.S} = 2^b\,\Gamma(\tfrac{a+b+9}{2}) \,/\, \Gamma(b)\,[\Gamma(\tfrac{b}{2}) \,/\, \Gamma(\tfrac{a+1}{2}) $$

$$ \{\frac{8a(-15+23a-9a^2+a^3+49b-14ab+21a^2b+7b^2+35ab^2+7b^3)}{(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)(a-b+7)}\} - \Gamma(\tfrac{b+1}{2}) \,/\, \Gamma(\tfrac{a}{2})\,\{\frac{16(15+49a-7a^2+7a^3+23b+14ab+35a^2b+9b^2+21ab^2+b^3)}{(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)(a-b+7)}\}] $$

$-2^b\,\Gamma(\frac{a+b+9}{2})$ / $\Gamma(b)$ [$\Gamma(\frac{b+1}{2})$ / $\Gamma(\frac{a}{2})$ $\{\frac{16(15+23a+9a^2+a^3+49b+14ab+21a^2b-7b^2+35ab^2+7b^3)}{(a-b-7)(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)}\}$

$-\Gamma(\frac{b}{2})/\Gamma(\frac{a+1}{2})\{\frac{8b(-15+49a+7a^2+7a^3+23b-14ab+35a^2b-9b^2+21ab^2+b^3)}{(a-b-7)(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)}\}$]

On simplification, we get

$${}_2F_1(a, b ; \frac{a+b+9}{2}; \frac{1}{2}) = 2^b\,\Gamma(\frac{a+b+9}{2})/(a-b)\,\Gamma(b)\,[\Gamma(\frac{b}{2})\,/\Gamma(\frac{a+1}{2})$$

$$\{\frac{8a(-15+23a-9a^2+a^3+49b-14ab+21a^2b+7b^2+35ab^2+7b^3)}{(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)(a-b+7)} + \frac{8b(-15+49a+7a^2+7a^3+23b-14ab+35a^2b-9b^2+21ab^2+b^3)}{(a-b-7)(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)}\}$$

$$-\ \Gamma(\frac{b+1}{2})\,/\Gamma(\frac{a}{2})\{\frac{16(15+49a-7a^2+7a^3+23b+14ab+35a^2b+9b^2+21ab^2+b^3)}{(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)(a-b+7)} + \frac{16(15+23a+9a^2+a^3+49b+14ab+21a^2b-7b^2+35ab^2+7b^3)}{(a-b-7)(a-b-5)(a-b-3)(a-b-1)(a-b+1)(a-b+3)(a-b+5)}\}]$$

Thus, we prove the result (12)

Similarly, we can prove the other results.

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