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HOMOMORPHISMS AND COMPOSITION OPERATORS ON WEIGHTED SPACES OF ANALYTIC FUNCTIONS

J. S. Manhas*
Department of Mathematics and Statistics Sultan Qaboos University, College of Science
P.O. Box 36, P.C. 123, Al-Khod, Muscat Sultanate of Oman
Email: manhas@squ.edu.om

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ABSTRACT

Let V be an arbitrary system of weights on an open connected subset G of $\mathbb{C}$. . Let $HV_b(G)$ and $HV_0(G)$ be the weighted locally convex spaces of analytic functions with a topology generated by seminorms which are weighted analogues of the supremum norm. In the present article, we characterize all continuous linear transformations on the spaces $HV_b(G)$ and $HV_0(G)$ which are composition operators. Also, for some system of weights, we have proved that homomorphisms on weighted algebras of analytic functions are composition operators.

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1. INTRODUCTION:

The theory of composition operators on spaces of analytic functions has a deep interaction with the theory of isometries and homomorphisms. De Leeuw, Rudin and Wermer [10] and Nagasawa [22] have described isometries of Hardy spaces $H^1(D)$ and $H^\infty(D)$ as a product of multiplication operators and composition operators. These composition operators have also made their appearance in the isometries of $H^p(\partial D)$ for $1 < p < \infty, p \neq 2$ [11] and isometries of Bergman spaces [15, 19]. In [7], Contreras and Hernandez-Diaz have made a study of weighted composition operators on Hardy spaces whereas Mirzakarimi and Siddighi [20] have considered these operators on Bergman and Dirichlet spaces. On Bloch and Block-type spaces, these operators are studied by MacCluer and Zhao [18], Ohno [23], Ohno and Zhao [26] and Ohno, Stroethoff and Zhao [24]. In [25], Ohno and Takagi have obtained some properties of these operators on the disc algebra and the Hardy space $H^\infty(D)$. Also, recently, Montes-Rodriguez [21] and Contreras and Hernandez-Diaz [6] have studied the behaviour of these operators on weighted Banach spaces of analytic functions. The applications of these operators can be found in the theory of semigroups and dynamical systems (see [14] and [31]).

As we know, three monographs (see Cowen and MacCluer [9], Shapiro [28] and Singh and Manhas [29]) speak the volume of work done on composition operators on spaces of analytic functions, the theory of composition operators is still to be explored much more in many directions on these different spaces of analytic functions. In the present paper, we present a study of composition operators and homomorphisms on the weighted locally convex spaces of analytic functions $HV_b(G)$ and $HV_0(G)$ for different nice systems of weights V on an open connected subset G of $\mathbb{C}$. .
We have organized this paper into four sections. The preliminaries required for proving the results in the remaining sections are reported in Section 2. In the third section we present characterizations of composition operators on the spaces $HV_0(G)$ and $HV_b(G)$ generalising the results of [5]. In Section 4, we have explored the system of weights for which homomorphisms on these spaces are composition operators

2. PRELIMINARIES:

Let $H(G)$ be the space of all analytic functions from an open connected subset G of the complex plane $\mathbb{C}$. into $\mathbb{C}$. and let $H^\infty(G)$ be the space of bounded analytic functions on G . By an **arbitrary system** of weights on G , we mean a directed

Corresponding author: J. S. Manhas*, Email: manhas@squ.edu.om

upward set V of non-negative upper semicontinuous functions on G such that for each $z \in G$, there exists $v \in V$ for which $v(z) > 0$. Let U and V be two arbitrary systems of weights on G. Then we say that $U \le V$ if for each $u \in U$, there exists $v \in V$ such that $u \le v$. We write $U \cong V$ if $U \le V$ and $V \le U$. Now, we can define the weighted spaces of analytic functions as

$$HV_b(G) = \{f \in H(G) : vf \text{ is bounded on } G, \text{ for each } v \in V\}$$

and

$$HV_0(G) = \{f \in H(G) : vf \text{ vanishes at infinity on } G, \text{ for every } v \in V\}.$$

For $v \in V$ and $f \in H(G)$, if we define $\|f\|_v = \sup\{v(z)|f(z)| : z \in G\}$, then $\|\cdot\|_v$ can be regarded as a seminorm on either $HV_b(G)$ or $HV_0(G)$.Clearly the family $\{\|\cdot\|_v : v \in V\}$ of seminorms defines a Hausdorff locally convex topology on each of these spaces and with this topology the vector spaces $HV_0(G)$ and $HV_b(G)$ are called the weighted locally convex spaces of analytic functions. The family of closed absolutely convex neighbourhoods of the form $B_v = \{f \in HV_b(G)(resp.HV_0(G)) : \|f\|_v \le 1\}$ is a basis of these spaces. Throughout the article, we assume that for each $z \in G$, there exists $f_z \in HV_0(G)$ for which $f_z(z) \neq 0$. Also, if $U \le V$, then clearly $HV_0(G) \subseteq HU_0(G)$ and $HV_b(G) \subseteq HU_b(G)$. Also, we define the product of U and V as $UV = \{u.v : u \in U \text{ and } v \in V\}$. It is easy to see that UV is again a system of weights on G. Now, if $V \le V^2$, then it can be easily seen that $HV_b(G)$ is a weighted topological algebra.

Next, we define some more systems of weights using the definitions of weights given in ([2],[4],[5]).

Let $v \in V$. Then we define $\tilde{w} : G \to \mathbb{R}^+$ as $\tilde{w}(z) = \sup\{|f(z)| : \|f\|_v \le 1\} \le \frac{1}{v(z)}$ and $\tilde{v}(z) = 1/\tilde{w}(z)$, for every $z \in G$. In case $\tilde{w}(z) \neq 0$, for all $z \in G, \tilde{v}$ is an upper semicontinuous and we call it an associated weight of v. Let $\tilde{V}$ denote the system of all associated weights of V. An arbitrary system of weights V is called a **reasonable** system if it satisfies the following properties:

(2.a) for each $v \in V$, there exists $\tilde{v} \in \tilde{V}$ such that $v \le \tilde{v}$.

(2.b) for each $v \in V$, $\|f\|_v \le 1$ if and only if $\|f\|_{\tilde{v}} \le 1$, *for* every $f \in H(G)$.

(2.c) if $v \in V$, then for every $z \in G$, there exists $f_z \in B_v$ such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$.

Let $v \in V$. Then v is called **essential** if there exists a constant $\alpha > 0$ such that $v(z) \le \tilde{v}(z) \le \alpha v(z)$, for each $z \in G$. If each weight $v \in V$ is essential, then we call V as an **essential system** of weights on G and hence $V \cong \tilde{V}$. For example, let $G = D$, the open unit disc and let $f \in H(D)$ be non-zero. Then define $v_f(z) = M(f, r)^{-1}$, where $M(f, r) = \sup\{|f(z)| : |z| = r\}$. Clearly v_f is a weight satisfying $\tilde{v}_f = v_f$ and thus the family $V = \{v_f : f \in H(D), f \text{ is non-zero}\}$ is an essential system of weights on D. For more details on the weighted Banach spaces of analytic functions and the weighted locally convex spaces of analytic functions associated with these weights, we refer to ([1], [2], [3], [4], [5], [12], [16], [17]). For basic definitions and facts in complex analysis and functional analysis, we refer to ([8], [13], [27]).

Let $L(G)$ be the vector space of all scalar-valued functions on G and let $F(G)$ be a topological vector space of scalar-valued analytic functions on G. Let $\varphi : G \to G$. Then define the linear map $C_\varphi : F(G) \to L(G)$ as $C_\varphi(f) = f \circ \varphi$, for every

$f \in F(G)$. In case C_φ takes $F(G)$ into itself and is continuous, we call C_φ, **the composition operator** on $F(G)$ induced by the symbol φ.

3. ANALYTIC FUNCTIONS INDUCING COMPOSITION OPERATORS:

Every self-analytic map $\varphi : D \longrightarrow D$ induce a composition operator on the Hardy space $H^\infty(D)$. But these maps do not necessarily induce composition operators on the weighted space $H_v^\infty(D)$, for general weights v. For example (see[5]), consider the weight $v(z) = e^{-(1-|z|)^{-1}}$, for $z \in D$. Then $v = \tilde{v}$. Let $\varphi : D \longrightarrow D$ be defined as $\varphi(z) = (z+1)/2$, for every $z \in D$. Then for $z=r\in\mathbb{R}$, we have $\frac{v(z)}{v(\varphi(z))} = \frac{v(r)}{v(\varphi(r))} = re^{\frac{1}{1-r}}$, for $0 < r < 1$. Then as $r \longrightarrow 1$, $\frac{v(r)}{v(\varphi(r))} \longrightarrow \infty$, so C_φ is not bounded on $H_v^\infty(D)$.

In this section we characterize composition operators on the spaces $HV_b(G)$ and $HV_0(G)$. We begin with the following straightforward theorem.

Theorem: 3.1 Let U and V be arbitrary systems of weights on G. Let $\varphi \in H(G)$ be such that $\varphi(G) \subseteq G$. Then $C_\varphi : HU_b(G) \to HV_b(G)$ is a composition operator if $V \le U \circ \varphi$.

Remark: 3.1 The condition $V \le U \circ \varphi$ in the above theorem is not a sufficient condition for C_φ to be a composition operator from $HU_0(G) \to HV_0(G)$. For instance, let $G = \{z \in \mathbb{C}: z = x + iy, x = x + iy, x > 0\}$ be the right half plane. Let $U = V$ be the system of constant weights on G. Let $\varphi : G \to G$ be defined as $\varphi(z) = z_0$, for every $z \in G$, where $z_0 \in G$ is fixed. Then clearly the inequality $V \le U \circ \varphi$ is true. But $C_\varphi : HU_0(G) \to HV_0(G)$ is not even an into map. For instance, if we take $f(z) = 1/z$, for every $z \in G$, then $f \in HU_0(G)$ but $C_\varphi(f) \notin HV_0(G)$. So, in order to show that $C_\varphi : HU_0(G) \to HV_0(G)$ is a composition operator, we need an additional condition on φ. Let $v \in V$ and $\varepsilon > 0$. Then consider the set $F(v,\varepsilon) = \{z \in G : v(z) \ge \varepsilon\}$. Clearly $F(v,\varepsilon)$ is a closed subset of G. In the next theorem we have obtained a sufficient condition for C_φ to be composition operator from $HU_0(G)$ into $HV_0(G)$.

Theorem: 3.2 Let U and V be arbitrary systems of weights on G. Let $\varphi \in H(G)$ be such that $\varphi(G) \subseteq G$. Then $C_\varphi : HU_0(G) \to HV_0(G)$ is a composition operator if

(i) $V \le U \circ \varphi$;
(ii) for every $v \in V$, $\varepsilon > 0$ and compact set $K \subseteq G$, the set $\varphi^{-1}(K) \cap F(v,\varepsilon)$ is compact.

Proof: In view of Theorem 3.1, condition (i) implies that $C_\varphi : HU_b(G) \to HV_b(G)$ is a composition operator. To show that $C_\varphi : HU_0(G) \to HV_0(G)$ is a composition operator, it is enough to prove that C_φ in an into map. For, let $f \in HU_0(G)$. Let $v \in V$ and $\varepsilon > 0$. Then we consider the set $K = \{z \in G : v(z)|f(\varphi(z))| \ge \varepsilon\}$. We shall show that K is a compact subset of G. By condition (i), there exists $u \in U$ such that $v(z) \le u(\varphi(z))$, for every $z \in G$. Let $S = \{z \in G : u(z)|f(z)| \ge \varepsilon\}$. Then clearly S is a compact subset of G and $\varphi(K) \subseteq S$. Let $M = \sup\{|f(z)| : z \in S\}$. Then $M > 0$ and $S \subseteq F(u, \varepsilon/M)$. By condition (ii), the set $\varphi^{-1}(S) \cap F(v, \varepsilon/M)$ is compact. Since K is a closed subset of the set $\varphi^{-1}(S) \cap F(v, \varepsilon/M)$, it follows that K is compact. Thus $C_\varphi(f) \in HV_0(G)$. This completes the proof.

Corollary: 3.1 Let U and V be arbitrary systems of weights on G. Let $\varphi \in H(G)$ be such that $\varphi(G) \subseteq G$. Then

(i) $C_\varphi : HU_b(G) \to HV_b(G)$ is a composition operator if $V \leq U \circ \varphi$.

(ii) $C_\varphi : HU_0(G) \to HV_0(G)$ is a composition operator if $V \leq U \circ \varphi$ and φ is a conformal mapping of G onto itself.

The converse of the above corollary may not be true. That is, if C_φ is a composition operator on $HV_b(G)$ and $HV_0(G)$ then $\varphi \in H(G)$ may not be conformal mapping of G onto itself. For example, let $V = \{\lambda \chi_K : \lambda \geq 0, K \subseteq G, K \text{ is compact}\}$, then it can be easily seen that C_φ is a composition operator on $HV_0(G)$ if and only if $\varphi : G \longrightarrow G$ is an analytic map.

In the next theorem, we obtained a necessary and sufficient condition for C_φ to be a composition operator on $HV_b(G)$ in terms of the inducing map φ and the system of weights V.

Theorem: 3.3 Let V be an arbitrary system of weights on G and let U be a reasonable system of weights on G. Let $\varphi \in H(G)$ be such that $\varphi(G) \subseteq G$. Then $C_\varphi : HU_b(G) \to HV_b(G)$ is a composition operator if and only if $V \leq \tilde{U} \circ \varphi$.

Proof: Suppose that $C_\varphi : HU_b(G) \to HV_b(G)$ is a composition operator. Let $v \in V$. Then by the continuity of C_φ at the origin, there exists $u \in U$ and a neighbourhood B_u of the origin in $HU_b(G)$ such that $C_\varphi(B_u) \subseteq B_v$. Let $\tilde{u}$ be the associated weight of u. Then $\tilde{u} \in \tilde{U}$. Now, we claim that $v \leq \tilde{u} \circ \varphi$. Fix $z_0 \in G$. Then by (2.c), there exists $f_0 \in B_u$ such that $|f_0(\varphi(z_0))| = 1/\tilde{u}(\varphi(z_0))$. Further, it implies that $C_\varphi(f_0) \in B_v$. That is, $v(z)|f_0(\varphi(z))| \leq 1,$ for every $z \in G$. In particular, for $z = z_0$, we have $v(z_0) \leq \tilde{u}(\varphi(z_0))$. This proves our claim and hence $V \leq \tilde{U} \circ \varphi$.

Conversely, suppose that the condition is true. To show that $C_\varphi : HU_b(G) \to HV_b(G)$ is a composition operator, it is sufficient to show that C_φ is continuous at the origin. For, let $v \in V$ and B_v be a neighbourhood of the origin in $HV_b(G)$. Then by the given condition, there exists $\tilde{u} \in \tilde{U}$ with $u \in U$ such that $v \leq \tilde{u} \circ \varphi$. That is, $v(z) \leq \tilde{u}(\varphi(z)),$ for every $z \in G$. Now we claim that $C_\varphi(B_u) \subseteq B_v$. Let $f \in B_u$. Then by (2.b), $\|f\|_u \leq 1$ if and only if $\|f\|_{\tilde{u}} \leq 1$. Now

$$\begin{aligned} \|C_\varphi f\|_v &= \sup\{v(z)|f(\varphi(z))| : z \in G\} \\ &\leq \sup\{\tilde{u}(\varphi(z))|f(\varphi(z))| : z \in G\} \\ &\leq \sup\{\tilde{u}(z)|f(z)| : z \in G\} = \|f\|_{\tilde{u}} \leq 1. \end{aligned}$$

This proves that $C_\varphi f \in B_v$ and hence C_φ is a composition operator. This completes the proof of the theorem.

Corollary: 3.2 Let U and V be reasonable systems of weights on G. Let $\varphi \in H(G)$ be such that $\varphi(G) \subseteq G$. Then the following statements are equivalent:

(i) $C_\varphi : HU_b(G) \to HV_b(G)$ is a composition operator;

(ii) $V \leq \tilde{U} \circ \varphi$;

(iii) $\tilde{V} \leq \tilde{U} \circ \varphi$.

Corollary: 3.3 Let V be an arbitrary system of weights on G and let U be an essential system of weights on G. Let $\varphi \in H(G)$ be such that $\varphi(G) \subseteq G$. Then $C_\varphi : HU_b(G) \to HV_b(G)$ is a composition operator if and only if $\mathbf{V} \leq \mathbf{U} \circ \varphi$.

Proof: Follows from Theorem 3.3 since $U \approx \tilde{U}$. □

Theorem: 3.4 Let V be an arbitrary system of weights on G and let U be an essential system of weights on G such that each weight of V and U vanishes at infinity. Let $\varphi \in H(G)$ be such that $\varphi(G) \subseteq G$. Then $C_\varphi : HU_0(G) \to HV_0(G)$ is a composition operator if and only if $V \leq U \circ \varphi$.

Proof: Follows from Theorem 3.2 and Corollary 3.3. □

Example: 3.1 Let $G = D$, the open unit disc and let v be a weight defined as $v(z) = 1 - |z|^2$, for every $z \in G$. Let $V = \{\lambda v : \lambda > 0\}$. Then clearly V is an essential system of weights on G. Let $\varphi : G \to G$ be an analytic map defined by $\varphi(z) = (z+1)/2$ for every $z \in G$. Now by the Pick-Schwarz Lemma, it follows that

$$\left(1-|z|^2\right)\left|\varphi'(z)\right| \leq 1 - \left|\varphi(z)\right|^2, \text{ for every } z \in G.$$

That is, $v(z) \leq 2v\left(\varphi(z)\right)$, for every $z \in G$. Hence by Theorem 3.1, C_φ is a composition operator on $HV_b(G)$.

Remark: 3.2. If $G = D$ and U and V consist of single continuous weights only, then Corollary 3.2, Corollary 3.3 and Theorem 3.4 reduces to the results of Proposition 2.1 and Corollary 2.2 of [5].

4. HOMOMORPHISMS AND COMPOSITION OPERATORS:

We shall begin with a characterization of all continuous linear operators on $HV_b(G)$ which are composition operators and this parallels a standard result for functional Hilbert spaces.

For each $z \in G$, the point evaluation δ_z defines a continuous linear functional on $HV_b(G)$. If we put $\Delta(G) = \{\delta_z : z \in G\}$, then $\Delta(G)$ is a subset of the continuous dual $HV_b(G)^*$.

Theorem: 4.1 Let $\Phi : HV_b(G) \longrightarrow HV_b(G)$ be a linear transformation. Then there exists $\varphi : G \longrightarrow G$ such that $\Phi = C_\varphi$ if and only if the transpose mapping Φ' from $HV_b(G)^*$ into the algebraic dual $HV_b(G)'$ leaves $\Delta(G)$ invariant. In case V is a reasonable system of bounded weights on G and $\Phi'(\Delta(G)) \subset \Delta(G)$, φ is necessarily analytic and $\Phi = C_\varphi$ is continuous if and only if $V \leq \tilde{V} o \varphi$.

Proof: Suppose that $\Phi = C_\varphi$ for some $\varphi : G \longrightarrow G$. Let $z \in G$ and $f \in HV_b(G)$. Then

$\left(\Phi'\delta_z\right)(f) = (\delta_z o \ \Phi)(f) = \delta_z(\Phi(f)) = \delta_z(C_\varphi f) = f(\varphi(z)) = \delta_{\varphi(z)}(f)$.

This implies that $\Phi' \delta_z = \delta_{\varphi(z)}$. Conversely, let us suppose that $\Phi'(\Delta(G)) \subset \Delta(G)$. For $z \in G$, if we define $\varphi(z)$ to be the unique element of G such that $\Phi' \delta_z = \delta_{\varphi(z)}$. Let $f \in HV_b(G)$. Then $\Phi(f)(z) = \delta_z(\Phi(f)) = (\delta_z o \ \Phi)(f) = \left(\Phi'\delta_z\right)(f) = \delta_{\varphi(z)}(f) = f(\varphi(z)) = C_\varphi(f)(z)$. Thus $\Phi = C_\varphi$. Also, since the identity function $f(z) = z$ belongs to $HV_b(G)$ and the range of C_φ is contained in $H(G)$, φ is necessarily an analytic map. Also, in view of Theorem 3.3, $\Phi = C_\varphi$ is continuous when $V \leq \tilde{V} o \varphi$. □

Theorem: 4.2. Let G be an open connected bounded subset of $\mathbb{C}$ and let V be a system of bounded weights on G such that $V \leq V^2$. Let $HV_b(G) \longrightarrow \mathbb{C}$ be a non-zero multiplicative linear functional. Then there exists $z_0 \in G$ such that $\Phi = \delta_{z_0}$.

Proof: Let $\lambda \in \mathbb{C}$ and let K_λ denote the constant function $K_\lambda(z) = \lambda$, for every $z \in G$. Since each weight $v \in V$ is bounded , it follows that each constant function $K_\lambda \in HV_b(G)$. Let $HV_b(G) \longrightarrow \mathbb{C}$ be a non-zero multiplicative linear functional . Then we have $\Phi(K_1) = \Phi(K_1.K_1) = \Phi(K_1)\Phi(K_1)$. That is , $\Phi(K_1)$ is equal to zero or one. In case $\Phi(K_1) = 0$, it follows that $\Phi(f) = \Phi(f.K_1) = \Phi(f)\Phi(K_1) = 0$, for every $f \in HV_b(G)$. Thus $\Phi = 0$, a contradiction. This shows that

$\Phi(K_1)=1$. Further, it implies that $\Phi(K_\lambda)=\Phi(K_\lambda.K_1)=\Phi(\lambda.K_1)=\lambda\Phi(K_1)=\lambda$. Let $f: G\longrightarrow \mathbb{C}$ be defined as $f(z)=z$, for every $z\in G$. Then Clearly $f\in HV_b(G)$. Now we fix $z_0=\Phi(f)$. We shall show that $z_0\in G$. Suppose that $z_0\notin G$. Then we define the function $h_{z0}:G\longrightarrow \mathbb{C}$ as $h_{z_0}(z)=\frac{1}{z-z_0}$, for every $z\in G$. Again, since each weight $v\in V$ is bounded and G is a bounded domain , it follows that $h_{z_0}\in HV_b(G)$. Also, from the definition of h_{z_0}, we have $(z-z_0)h_{z_0}(z)=1$, for every $z\in G$. That is , $(f(z)-K_{z_0}(z))h_{z_0}(z)=1$, for every $z\in G$. Thus $(f-K_{z_0})h_{z_0}=K_1$. Further, it implies that

$\Phi(K_1)=\Phi(f-K_{z_0})\Phi(h_{z_0})=(\Phi(f)-\Phi(K_{z_0}))\Phi(h_{z_0})=(z_0-z_0)\Phi(h_{z_0})=0$, which is a contradiction because $\Phi(K_1)=1$. This proves that $z_0\in G$. Now let $g\in HV_b(G)$. Then we define the function $h:G\longrightarrow \mathbb{C}$ as $h(z)=h_{z_0}(z)(g(z)-g(z_0))$, for $z\neq z_0$ and $h(z)=g^{/}(z_0)$, for $z=z_0$. It can be easily seen that $h\in HV_b(G)$. Now it readily follows that $(f-K_{z_0})h=g-K_{g(z_0)}$. Further, we have $\Phi((f-K_{z_0})h)=\Phi(g-K_{g(z_0)})$. That is $0=\Phi(g)-\Phi(K_{g(z_0)})$. Thus it follows that $\Phi(g)=g(z_0)=\delta_{z_0}(g)$. This proves that $\Phi=\delta_{z_0}$. With this the proof of the theorem is complete.

Theorem : 4.3. Let G_1 and G_2 be open connected bounded subsets of $\mathbb{C}$. Let V and U be systems of bounded weights on G_1 and G_2, respectively, such that $V\leq V^2$ and $U\leq U^2$. Let $\Phi: HV_b(G_1)\longrightarrow HU_b(G_2)$ be a non-zero algebra homomorphism. Then there exists a holomorphic map $\varphi: G_2\longrightarrow G_1$ such that $\Phi=C_\varphi$.

Proof: Since $HV_b(G_1)$ and $HU_b(G_2)$ contains constant functions, it follows that $K_1\in HV_b(G_1)$ and $\Phi(K_1)=\Phi(K_1).\Phi(K_1)$. Then using connectedness of G_2 , we can conclude that $\Phi(K_1)=K_1$. Further, it implies that $\Phi(K_\lambda)=K_\lambda$, for every $\lambda\in\mathbb{C}$. Now let $z_0\in G_2$. Then define $\delta_{z_0}: HV_b(G_1)\longrightarrow\mathbb{C}$ as $\delta_{z_0}(f)=(\Phi f)(z_0)$. Clearly δ_{z_0} is a multiplicative linear functional on $HV_b(G_1)$. Hence by Theorem 4.2, there exists $\alpha\in G_1$ such that $\delta_{z_0}(f)=\delta_\alpha(f)=f(\alpha)$, for every $f\in HV_b(G_1)$. Let $g: G_1\longrightarrow\mathbb{C}$ be defined as $g(z)=z$, for every $z\in G_1$. Then clearly $g\in HV_b(G_1)$ and $\delta_{z_0}(g)=g(\alpha)$. Thus it follows that $(\Phi g)(z_0)=\alpha$. Let us define $\varphi=\Phi(g)$. Thus $\varphi: G_2\longrightarrow G_1$ is an analytic map such that $(\Phi f)(z_0)=f(\alpha)=f(\Phi g)(z_0)=(fo\varphi)(z_0)$, $z_0\in G_2$. This shows that $\Phi(f)=C_\varphi(f)$, for every $f\in HV_b(G_1)$. Hence $\Phi=C_\varphi$. With this the proof of the theorem is complete. □

Theorem: 4.4. Let G be an open connected bounded subset of $\mathbb{C}$ and let V be a system of bounded weights on G such that $V\leq V^2$. Then a composition transformation C_φ on $HV_b(G)$ is invertible if and only if $\varphi: G\longrightarrow G$ is a conformal mapping.

Proof: If φ is a conformal mapping , then obviously C_φ is invertible on $HV_b(G)$. On the other hand, suppose A is the inverse of C_φ. Then we have $AC_\varphi.=C_\varphi A=I$. For f and g in $HV_b(G)$,we have $C_\varphi A(fg)=fg$. Further, it implies that $A(fg)o\varphi=fg=(C_\varphi Af).(C_\varphi Ag)=(Af)o\varphi.(Ag)o\varphi=(Af.Ag)o\varphi$. That is , $(A(fg)-Af.Ag)o\varphi=0$. Since C_φ is invertible, φ is nonconstant and hence the range of φ is an open set. Thus it follows that $A(fg)=Af.Ag$. According to Theorem 4.3, there exists an analytic map $\psi: G\longrightarrow G$ such that $A=C_\psi$. Let $f(z)=z$, for every $z\in G$. Then $f\in HV_b(G)$ and we have $(C_\psi C_\varphi f)(z)=(fo\varphi o\psi)(z)=(\varphi o\psi)(z)$, for every $z\in G$. Also, $(C_\varphi C_\psi)(z)=(fo\psi o\varphi)(z)=(\psi o\varphi)(z)$, for every $z\in G$. From this we conclude that φ is invertible with an analytic inverse map as ψ. Hence φ is a conformal mapping of G onto itself.

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