International Journal of Mathematical Archive-2(2), Feb. - 2011, Page: 241-245
Available online through www.ijma.info ISSN 2229 – 5046

SOME RESULTS ON σ-Lie ideals & GENERALIZED DERIVATIONS IN σ -PRIME RINGS

M. Rais Khan, Deepa Arora and M. Ali Khan

Department of Mathematics Jamia Millia Islamia(Central University) New Delhi-110025, India.
E-mail: *mohdrais_khan@yahoo.co.in, musk.deepa@gmail.com, mkhan91@gmail.com*

(Received on: 13-01-11; Accepted on: 30-01-11)

ABSTRACT

Let R be a 2-torsion free σ-prime ring with involution σ, U a non zero σ-Lie ideal and I a nonzero σ-ideal of R. Let F be a generalized derivation associated with a nonzero derivation d of R commuting with σ. In the present paper, we discuss the commutativity of a σ-prime ring R admitting a generalized derivation F satisfying any of the following properties:
(i) $d(x) \circ F(y) = 0$, (ii) $[d(x), F(y)]= 0$, (iii) $d(x) \circ F(y) \pm x \circ y = 0$, (iv) $[d(x), F(y)] \pm [x, y] = 0$, (v) $(d(x) \circ F(y)) \pm [x, y] = 0$,and (vi) $[d(x), F(y)] \pm x \circ y = 0$ for all x, y in an appropriate subset of R.

2000 Mathematics subject Classification: *16W10, 16W25, 16U80.*

Keywords: *Derivations, Lie ideals, rings with involution, σ-prime rings.*

1. INTRODUCTION:

Throughout, R will represent an associative ring with center Z(R). Recall that a ring R is prime if aRb = 0 implies a = 0 or b = 0. If R has an involution σ, then R is said to be σ-prime if aRb = aRσ (b) = 0 implies a = 0 or b = 0. Every prime ring equipped with an involution is σ -prime but the converse need not be true in general. As an example, taking $S = R\times R^0$ where R^0 is an opposite ring of a prime ring R with (x, y) = (y, x). Then S is not prime if (0, a) S (a, 0) = 0. But if we take (a, b) S (x, y) = 0 and (a, b) S σ ((x, y)) = 0, then aRx × yRb = 0 and aRy × xRb = 0, and thus aRx = yRb = aRy = xRb = 0. This shows that R is σ-prime (see for reference [7]). An ideal I of R is a σ-ideal if I is invariant under σ i.e. σ (I) = I. An additive subgroup U of R is said to be a Lie-ideal if [u, r] ∈ U for all u ∈ U and r ∈ R. A Lie-ideal U which satisfies σ (U) = U is called a σ-Lie ideal. If U is a Lie (resp. σ-Lie) ideal of R, then U is called a square closed Lie (resp. σ-Lie) ideal if $u^2 \in U$ for all u ∈ U. Oukhtite et al. defined a set of symmetric and skew symmetric elements of R as $Sa_\sigma(R) = \{x \in R \mid \sigma(x) = \pm x\}$. We need the basic commutator identities. For any x, y, z ∈ R,

- [x, yz] = y[x, z] + [x, y]z , [xy, z] = x[y, z] + [x, z]y
- x o (yz) = (x o y)z – y[x, z] = y(x o z) + [x, y]z
- (xy) o z = x(y o z) – [x, z]y = (x o z)y + x[y, z].

As defined by Bresar [4], an additive map F: R → R is called a generalized derivation associated with d if there exists a derivation d: R → R (an additive map d: R → R is called a derivation if d (xy) = d(x)y + xd(y) holds for all x, y ∈ R) such that F (xy) = F(x) y + xd(y) for all x, y ∈ R. One can easily check that the notion of generalized derivation covers the notions of a derivation and a left multiplier (i.e. F (xy) = F(x) y for all x, y ∈ R). Particularly, we can observe that: For a fixed a ∈ R, the map $d_a : R \to R$ defined by $d_a(x) = [a, x]$ for all x ∈ R is a derivation which is said to be an inner derivation. An additive map $g_{a,b}$: R → R is called a generalized inner derivation if $g_{a,b}(x) = ax + xb$ for some fixed a, b ∈ R. It is easy to see that if $g_{a,b}(x)$ is a generalized inner derivation, then

$g_{a,b}(x)\,(xy) = g_{a,b}(x)y + xd_{-b}(y)$ for all x, y ∈ R, where d_{-b} is an inner derivation.

A number of authors [1, 2, 3, 15, 16, 17, 18] have established an enormous theory concerning derivations and generalized derivations of prime rings and are still adding. In 2005, Oukhtite et al. gave an extension of prime rings in the form of σ -prime rings and proved numerous results which hold true for prime rings (see for references [7 - 14]). Huang too contributed to the newly emerged theory by extending the work namely of, Ashraf et al. and Rehman et al. [2] and [3] in [18] and [19] respectively. Further, in [5] and [6] author et al. extended results concerning derivations and generalized derivations of σ -prime rings to some more general settings. In the present paper we develop some more results in the same context.

In [1] , Ashraf et al. studied the commutativity of a prime ring R admitting a generalized derivation F with associated derivation d satisfying anyone of the following properties: (i) $d(x) \circ F(y) = 0$, (ii) $[d(x), F(y)] = 0$, (iii) $d(x) \circ F(y) \pm x \circ y = 0$, (iv) $(d(x) \circ F(y)) \pm [x, y] = 0$, (v) $[d(x), F(y)] \pm [x, y] = 0$, and (vi) $[d(x), F(y)] \pm x \circ y = 0$ for all x, y ∈ I, where I is non

***Corresponding author: Deepa Arora** *
***E-mail: musk.deepa@gmail.com**

zero ideal of prime ring R. In 2007, Huang [20] obtained similar results by considering Lie ideals instead. The objective of the paper is to extend these results for σ -prime rings.

A continuous approach in the direction of σ -prime rings is still on. However, results concerning inner derivations and generalized inner derivations in σ -prime rings are still naive.

2. PRELIMINARY RESULTS:

We begin with

Lemma 2.1([14, Lemma 4]) If $U \subseteq Z(R)$ is a σ-Lie ideal of a 2-torsion free σ-prime ring R and $a, b \in R$ such that $aUb = \sigma(a)Ub = 0$ or $aUb = aU\sigma(b)$, then $a = 0$ or $b = 0$.

Lemma 2.2([13, Theorem 1.1])Let R be a 2-torsion free σ-prime ring, U a nonzero Lie ideal of R and d a nonzero derivation of R which commutes with σ. If $d^2(U) = 0$,then $U \subset Z(R)$.

Lemma 2.3 ([13, Lemma 2.3]) Let $0 \neq U$ be a σ-Lie ideal of a 2-torsion free σ-prime ring R. If $[U, U] = 0$, then $U \subset Z(R)$.

Lemma 2.4 ([11, Lemma 2.2]) Let R be a 2-torsion free σ-prime ring and U a nonzero σ-Lie ideal of R. If d is a derivation of R which commutes with σ and satisfies $d(U) = 0$, then either $d = 0$ or $U \subseteq Z(R)$.

We prove the following Lemma

Lemma 2.5 Let R be a 2-torsion free σ-prime ring and U a nonzero σ-Lie ideal of R. If d is a derivation of R which commutes with σ such that $d(x) \circ y = 0$ for all $x, y \in U$, then either $d = 0$ or $U \subseteq Z(R)$.

Proof: Suppose that $U \not\subset Z(R)$. We have

$$d(x) \circ y = 0 \text{ for all } x, y \in U. \tag{1}$$

Replacing y by yz in (1), we get

$$y\,[d(x), z] = 0 \text{ for all } x, y, z \in U. \tag{2}$$

Again replacing y by $d^2(x)\, y$ in (2), we obtain

$$d^2(x)\, y\, [d(x), z] = 0 \text{ for all } x, y, z \in U$$

or

$$d^2(x)\, U\, [d(x), z] = 0 \text{ for all } x, z \in U. \tag{3}$$

Using similar techniques as used in the proof of the result [19, Theorem 3.2], we get $d = 0$.

3. MAIN RESULTS:

Theorem 3.1: Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose there exists a generalized derivation F associated with a nonzero derivation d, commuting with σ, such that $d(x) \circ F(y) = 0$, for all $x, y \in U$, then $U \subseteq Z(R)$.

Proof: Suppose $U \not\subset Z(R)$. We have

$$d(x) \circ F(y) = 0 \text{ for all } x, y \in U. \tag{4}$$

Replacing y by yz in (4) and using (4), we obtain

$$(d(x) \circ y)\, d(z) - y[d(x), d(z)] - F(y)[d(x), z] = 0 \text{ for all } x, y, z \in U. \tag{5}$$

Now replacing z by zd(x) in (5), yields

$$(d(x) \circ y))\, d(z\, d(x)) - y[d(x), z\, d(x)] - F(y)[d(x), z\, d(x)] = 0,$$

or

$$(d(x) \circ y))\{d(z)\, d(x) + z\, d^2(x)\} - y[d(x), z\,]d(x) - F(y)[d(x), z]\, d(x) = 0 \text{ for all } x, y, z \in U. \tag{6}$$

Using (5) in the above obtained relation (6), implies

$$(d(x) \circ y))\, z\, d^2(x) = 0 \text{ for all } x, y, z \in U.$$

or

$$(d(x) \circ y))\, U\, d^2(x) = 0 \text{ for all } x, y \in U \tag{7}$$

Let $x \in U \cap Sa_\sigma(R)$. Since d commutes with σ, application of Lemma 2.1 in (7), yields

$$d(x) \circ y = 0 \text{ or } d^2(x) = 0 \text{ for all } y \in U. \tag{8}$$

Let $x \in U$. Since $x - \sigma(x) \in U \cap Sa_\sigma(R)$, from (8) it follows that

$$d(x - \sigma(x)) \circ y = 0 \text{ or } d^2(x - \sigma(x)) = 0 \text{ for all } y \subset U.$$

Case 1: Suppose $d(x - \sigma(x)) \circ y = 0$; if $d(x + \sigma(x)) \circ y = 0$, then

$$(d(x) - d(\sigma(x)) + d(x) + d(\sigma(x))) \circ y = 0$$

or $2d(x) \circ y = 0$. Since char $R \neq 2$, we get $d(x) \circ y = 0$ for all $x, y \in U$.

Case 2: Now suppose that $d^2(x - \sigma(x)) = 0$. Since d commutes with σ, we have

$$d(d(x) - d(\sigma(x))) = 0$$

or

$$d^2(x) - d(\sigma(d(x))) = 0$$

or

$$d^2(x) = \sigma(d^2(x)).$$

Thus, $d^2(x) \in Sa_\sigma(R)$ and again by virtue of Lemma 2.1 and (7), we get

$$d(x) \circ y = 0 \text{ or } d^2(x) = 0.$$

In conclusion, for all $x \in U$ we have either $d(x) \circ U = 0$ or $d^2(x) = 0$. Accordingly, U is a union of two additive subgroups G and H, where

$G = \{x \in U \mid d(x) \circ y = 0, \text{ for all } y \in U\}$ and $H = \{x \in U \mid d^2(x) = 0\}$.

But a group cannot be a union of two of its subgroups and thus $U = G$ or $U = H$.

If $U = G$, then by virtue of Lemma 2.5 we get either $d = 0$ or $U \subseteq Z(R)$.

If $U = H$, then by application Lemma 2.2 we get either $d = 0$ or $U \subseteq Z(R)$.

Hence, both the cases yields either $d = 0$ or $U \subseteq Z(R)$. But, since d is non zero, then U must be contained in Z(R).

Theorem 3.2: Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose there exists a generalized derivation F associated with a nonzero derivation d, commuting with σ, such that $[d(x), F(y)] = 0$, for all x, y $\in$ U, then $U \subseteq Z(R)$.

Proof: Suppose $U \not\subset Z(R)$. We have

$$[d(x), F(y)] = 0 \text{ for all } x, y \in U. \quad (9)$$

Replacing y by yz in (9) and using (9), we obtain

$$F(y)[d(x), z] + [d(x), y]\, d(z) + y[d(x), d(z)] = 0 \quad (10)$$

Now replacing z by zd(x) in (10), yields

$$[d(x), y]\, z\, d^2(x) + y\, [d(x), zd^2(x)] = 0 \quad (11)$$

Substituting y by wy in (11) and applying (11), we have

$$[d(x), w]\, yz\, d^2(x) = 0$$

or $\quad [d(x), w]\, U\, U\, d^2(x) = 0$ for all $x, w \in U \quad (12)$

Let $x \in U \cap Sa_\sigma(R)$. Since d commutes with σ, application of Lemma 2.1 in (12), yields

$$[d(x), w] = 0 \text{ or } U\, d^2(x) = 0 \text{ for all } w \in U.$$

Let $x \in U$. Since $x - \sigma(x) \in U \cap Sa_\sigma(R)$, from (12) in combination with Lemma 2.1, it follows
$[d(x - \sigma(x)), w] = 0$ or $d^2(x - \sigma(x)) = 0$ for all $w \in U$.

Case 1: Suppose $[d(x - \sigma(x)), w] = 0$

or $\quad [d(x), w] = [d(\sigma(x)), w]$

$\quad\quad = \sigma([d(x), w])$

Hence by (12) we get

$$[d(x), w] = 0 \text{ or } d^2(x) = 0 \text{ for all } x \in U.$$

Case 2: Suppose $d^2(x - \sigma(x)) = 0$.

Using the same arguments as used in proof of Case 2 in Theorem 3.1, we have

$$[d(x), w] = 0 \text{ or } d^2(x) = 0 \text{ for all } x \in U. \quad (13)$$

Applying similar arguments as used in the proof of [19, Theorem 3.2] from (13), we conclude that d = 0. Since d is given to be non zero, a contradiction is obtained. Thus, U must be contained in Z(R).

Note: The technique used in proof of Theorem 3.2 from equation (12) onwards till end would be required further in the paper. We name this portion as result A.

Theorem 3.3: Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose R admits a generalized derivation F associated with a derivation d, commuting with σ, such that $d(x) o F(y) \pm x o y = 0$, for all x, y $\in$ U. If F = 0 or $d \neq 0$ then $U \subseteq Z(R)$.

Proof: (i) If F = 0, then $x o y = 0$ for all x, y $\in$ U $\quad$ (14)

For any $z \in U$ replacing y by yz in (14), we get

$x o yz = 0$

or $\quad (x o y)z - y[x, z] = 0$

or $\quad y [x, z] = 0$

or $\quad U[x, z] = 0$

or $\quad 1\, U[x, z] = 0$

or $\quad \sigma(1)\, U\, [x, z] = 0$, since U is σ-Lie idel Lie ideal of R,

we have

$$[x, z] = 0 \text{ for all } x, z \in U. \quad (15)$$

In view of Lemma 2.3, (15) implies that $U \subseteq Z(R)$.

Now assume that $d \neq 0$. We have

$$d(x) o F(y) = x o y \text{ for all } x, y \in U. \quad (16)$$

For any $z \in U$, replacing y by yz in (16), we get

$$(d(x) o y)\, d(z) - y\, [d(x), d(z)] - F(y)[d(x), z] + y[x, z] = 0 \quad (17)$$

Replacing z by zd(x) in (17) and applying (17), we obtain

$$(d(x) o y)\, z\, d^2(x) - y[d(x), zd^2(x)] + yz\, [x, d(x)] = 0 \quad (18)$$

Substitute y by zy in (18) and using (18), we have

$$[d(x), z]\, U\, d^2(x) = 0 \text{ for all } x, z \in U. \quad (19)$$

Consequently, by application of result A in (19), we get

$$U \subseteq Z(R).$$

(ii) Using similar techniques as used in (i), one can also prove the following theorem.

"Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose R admits a generalized derivation F associated with a derivation d, commuting with σ, such that $d(x) o F(y) + x o y = 0$, for all x, y $\in$ U. If F = 0 or $d \neq 0$ then $U \subseteq Z(R)$."

Theorem 3.4: Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose R admits a generalized derivation F associated with a derivation d, commuting with σ, such that $[d(x), F(y)] \pm [x, y] = 0$, for all x, y $\in$ U, then $U \subseteq Z(R)$.

Proof: (i) Suppose $U \not\subset Z(R)$. We have

$[d(x), F(y)] - [x, y] = 0$ for all $x, y \in U$. (20)

Replacing y by yz in (20), we get

$[d(x), F(y)]z + F(y)[d(x), z] + [d(x), y]d(z) + y[d(x), d(z)] = y[x, z] + [x, y]z$

or $F(y)[d(x), z] + [d(x), y]d(z) + y[d(x), d(z)] = y[x, z]$ for all $x, y, z \in U$.(using 20) (21)

Substituting z by zd(x) in (21) and applying (21), we obtain

$[d(x),y]zd^2(x)+yz\ [d(x), d^2(x)]+y[d(x),z]d^2(x)–yz[x,d(x)]=0$ (22)

Replacing y by wy in (22) and using (22), we have

$[d(x), w]\, yz\, d^2(x) = 0$ for all $x, y, z, w \in U$

or $[d(x), w]\, U\, d^2(x) = 0$ for all $x, w \in U$. (23)

Hence by application of result A we conclude the result.

(ii) If $[d(x), F(y)] + [x, y] = 0$ for all $x, y \in U$, then by using the similar technique with necessary variations, the result follows.

Theorem 3.5 Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose R admits a generalized derivation F associated with a derivation d, commuting with σ, such that $d(x) \circ F(y) \pm [x, y] = 0$, for all $x, y \in U$, then $U \subseteq Z(R)$.

Proof: (i) Suppose $U \not\subset Z(R)$. We have

$d(x) \circ F(y) - [x, y] = 0$, for all $x, y \in U$ (24)

Replacing y by yz in (24) and using (24) ,we get

$F(y)\ [d(x), z] + (d(x) \circ y)\ d(z) – y[d(x), d(z)] = y[x, z]$ (25)

Substituting z by zd(x) in (25) and applying (25), we obtain

$(d(x) \circ y)\, zd^2(x) –yz\ [d(x),d^2(x)]–y\ [d(x),z]\ d^2(x) = yz\ [x, d(x)]$ (26)

For any $w \in U$, replacing y by wy in (26) and using (26), we have

$[d(x), w]\, y\, z\, d^2(x) = 0$ for all $x, y, z \in U$

or $[d(x), w]\, U\, d^2(x) = 0$ for all $x, w \in U$. (27)

Reasoning as used in proof of result A, we conclude the result.

(ii) Using the similar technique as used in (i) with necessary variations, the result follows.

Theorem 3.6 Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose R admits a generalized derivation F associated with a derivation d, commuting with σ, such that $[d(x), F(y)] \pm x \circ y = 0$, for all $x, y \in U$, then $U \subseteq Z(R)$.

Proof: (i) Suppose $U \not\subset Z(R)$. We have

$[d(x) , F(y)] = x \circ y$, for all $x, y \in U$ (28)

Replacing y by yz in (28) and using (28), we get $F(y)\ [d(x), z] + [d(x), y]\ d(z) + y[d(x), d(z)] = - y[x, z]$ (29)

Substituting z by zd(x) in (29) and applying (29), we obtain

$yz\ [d(x), d^2(x)] + y\ [d(x), z]\ d^2(x) + [d(x), y]\ z\ d^2(x) = - yz\ [x, d(x)]$ (30)

For any $w \in U$, replacing y by wy in (30) and using (30), we have

$[d(x), w]\, y\, z\, d^2(x) = 0$ for all $x, y, z \in U$

or $[d(x), w]\, U\, d^2(x) = 0$ for all $x, w \in U$. (31)

Applying result A, we get the desired result.

(ii) Using similar arguments as used in (i) with necessary changes, the result follows in case $[d(x), F(y)] + x \circ y = 0$.

In view of these results we get the following corollary:

Corollary 3.7: Let R be a 2-torsion free σ-prime ring and I a nonzero σ-ideal of R. Suppose that R admits a generalized derivation F associated with a nonzero derivation d, commuting with σ, such that any of the following holds:

(i) $d(x) \circ F(y) = 0$,
(ii) $[d(x),F(y)] = 0$,
(iii) $d(x) \circ F(y) = x \circ y$,
(iv) $d(x) \circ F(y) \pm x \circ y = 0$,
(v) $[d(x), F(y)] \pm [x, y] = 0$ for all $x, y \in I$.

Then R is commutative.

REFERENCES:

[1] Ashraf, M, Ali, A, and Rani, R, "On generalized derivations of prime rings," *Southeast Asian Bulletin of Mathematics*, (4) **29** 2005, 669–675.

[2] Ashraf, M, Ali, A and Ali, S, Some commutativity theorems for rings with generalized derivations, Southeast Asian Bull. Math., **31**(2007), 415-421.

[3] Ashraf, M and Rehman, N, On commutativity of rings with derivations, Result. Math., **42** (2002), 3 – 8.

[4] Bresar, M, On the distance of the compositions of two derivations to the generalized derivations, Glasgow Math. J., (1) **33** (1991), 89-93.

[5] Khan, M.R , Arora, D and Khan, M.A, Remarks on Derivations of σ -prime rings, International Journal of Algebra, (16) **4** (2010), 761 – 767.

[6] Khan, M.R , Arora, D and Khan, M.A, Notes on derivations and Lie ideals in σ -prime rings, Advances in Algebra, (1) **3** (2010), 19–23.

[7] Oukhtite, L and Salhi, S, On generalized derivations of σ-prime rings, African Diaspora J. Math., (1) **5** (2006), 19-23.

[8] Oukhtite, L and Salhi, S, On commutativity of σ-prime rings, Glasnik Mathematicki, **41**(2006), 57-64.

[9] Oukhtite, L and Salhi, S, On derivations in σ-prime rings, Int. J. Algebra, **1**(2007), 241-246.

[10] Oukhtite, L and Salhi, S, Derivations and commutativity of σ-prime rings, Int. J. Contemp, **1**(2006), 439-448.

[11] Oukhtite, L and Salhi, S, σ-Lie ideals with derivations as homomorphisms and antihomomorphisms, Int. J. Algebra, **1**(2007), 235-239.

[12] Oukhtite, L and Salhi, S and Taoufiq, L, On generalized derivations and commutativity in σ- prime rings, Int. J. Algebra, **1**(2007), 227-230.

[13] Oukhtite, L and Salhi, S, Lie ideals and derivations of σ-prime rings, Int. J. Algebra, **1**(2007), 25-30.

[14] Oukhtite, L and Salhi. S, Centralizing automorphisms and Jordan Left derivations on σ- prime rings, Advances in Algebra, (1) **1**(2008), 19-26.

[15] Posner, E.C, Derivations in prime rings, Proc. Amer. Math. Soc., **8** (1957), 1093-1100.

[16] Quadri, M. A., Khan, M. S and Rehman, N, Generalized derivations and commutativity of prime rings, Indian J. Pure Appl. Math., 9 **34** (2003), 1393-1396.

[17] Rehman, N, Omary, R.M and Haetinger, C, On Lie structure of prime rings with generalized (α,β)–derivations, Bol. Soc. Paranaense Math., (2) **27** (2009), 43-52.

[18] Shuliang, H, Generalized derivations of σ-prime rings, Int. J. of Algebra, (18) **2** (2008), 867-873.

[19] Shuliang, H, Lie ideals and derivations of σ-prime rings, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math., (1) **17**(2010), 87 – 92.

[20] Shuliang, H, Generalized Derivations of Prime Rings, International Journal of Mathematics and Mathematical Sciences, (2007), Article ID 85612, 6 pages.

