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# COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF GENERALIZED SAKAGUCHI TYPE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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***ABSTRACT***

*In this paper we introduce a class* $K_s(\mathrm{A,B,s,t})$. *Further we obtain coefficient inequality for this class.*

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## 1. INTRODUCTION:

Let A denote the class of analytic and univalent functions f in $\Delta = \{z \in C : |z| < 1\}$ of the form,

$$f(\mathrm{z}) = \mathrm{z} + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and $S_s^*$ be the subclass of A consisting of univalent functions. For two functions $\mathrm{f, g} \in \mathrm{A}$, we say that the function f(z) is subordinate to g(z) in $\Delta$ and write $f \prec \mathrm{g}$ or $f(\mathrm{z}) \prec \mathrm{g(z)}$, if there exists an analytic function w(z) with w(0) = 0 and $|\mathrm{w(z)}| < 1$ $(\mathrm{z} \in \Delta)$, such that $f(\mathrm{z}) = \mathrm{g(w(z))}, (\mathrm{z} \in \Delta)$. In particular, if the function g is univalent in $\Delta$, the above subordination is equivalent to $f(0) = \mathrm{g}(0)$ and $f(\Delta) \subset g(\Delta)$.

Here we studied a Generalized Sakaguchi type class $S_s^*(\alpha, s, t)$.The function $f(\mathrm{z}) \in \mathrm{A}$ is said to be in the class $S_s^*(\alpha, s, t)$ if it satisfies,

$$\mathrm{Re}\left\{\frac{(s-t)zf'(z)}{f(sz) - f(zt)}\right\} > \alpha; \tag{2}$$

for some $\alpha \in [0,1), s, t \in C \; with \; t \neq s$, and for all $\mathrm{z} \in \Delta$. The class $S_s^*(\alpha, 1, t)$ was introduced and studied by Owa et al. [6] and when we take $\mathrm{t} = -1$ in above class, the class reduces in to $S_s^*(\alpha, 1, -1) = S_s^*(\alpha)$ which was introduced by Sakaguchi [4] and is called Sakaguchi Function of Order $\alpha$, see [8,9], where as $S_s^*(0) = S_s^*$ of Starlike Functions with respect to symmetrical point in $\Delta$.

Let $S_s^*(A, B, s, t)$, denote the class of functions of the form (1) and satisfying the condition,

$$\frac{(s-\mathrm{t})zf'(\mathrm{z})}{f(\mathrm{sz}) - f(\mathrm{zt})} \prec (1 + \mathrm{Az})/(1 + \mathrm{Bz}); \quad for\; s, \mathrm{t} \in \mathrm{C}, with\; \mathrm{t} \neq s \text{, where } (-1 \leq \mathrm{B} < \mathrm{A} \leq 1). \tag{3}$$

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In this paper, we consider be the class $K_s(\mathrm{A},\mathrm{B},\mathrm{s},\mathrm{t})$ of functions of the form (1) and satisfying the condition

$$\frac{(s-\mathrm{t})zf'(\mathrm{z})}{g(\mathrm{sz})-g(\mathrm{zt})} \prec \frac{(1+\mathrm{Az})}{(1+\mathrm{Bz})};\ for\,\mathrm{s,t}\in C\ with\ \mathrm{t}\neq s,\, g\in S_s^*(A,B,s,t), -1\leq \mathrm{B}<\mathrm{A}\leq 1, z\in\Delta. \tag{4}$$

By the definition of subordination it follows that if $f\in K_s(A,B,s,t)$ if and only if

$$\frac{(s-\mathrm{t})zf'(\mathrm{z})}{g(\mathrm{sz})-g(\mathrm{zt})} = \frac{1+\mathrm{Aw(z)}}{1+B\mathrm{w(z)}} = P(z);\ \left(s,t\in C, \mathrm{t}\neq s, g\in S_s^*(A,B,s,t), \left|w(z)\right|<1, w\in\Delta\right) \tag{5}$$

where

$$P(\mathrm{z}) = 1+\sum_{n=1}^{\infty} p_n z^n. \tag{6}$$

In the present paper, we obtain the coefficient estimate of the class $K_s(A,B,s,t)$ .

## 2. PRELIMINARY RESULT

To prove our main results, we need the following result:

**Lemma 2.1:** *If* $P(z)$ is given by (6), then

$$|\,p_n\,| \leq (A-B) \quad \text{(See [3]).} \tag{7}$$

**Lemma 2.2:** Let $f\in S_s^*(A,B,s,t)$, then $n\geq 1$,

$$|\,a_n\,| \leq \frac{(A-B)}{\left|(n-u_n)\right|}\left[1+(A-B)\sum_{i=2}^{n-1}\frac{|\,u_i\,|}{|\,(i-u_i)\,|} + (A-B)^2\sum_{i_2>i_1}^{n-1}\ \sum_{i_1=2}^{n-2}\frac{|\,u_{i_1}u_{i_2}\,|}{|\,(i_1-u_{i_1})\,||\,(i_2-u_{i_2})\,|} + .\ \ +(A-B)^{n-2}\prod_{i=2}^{n-1}\frac{|\,u_i\,|}{|\,(i-u_i)\,|}\right], \tag{8}$$

(See [10]), where $u_i = \dfrac{(s^i - t^i)}{(s-t)}$.

## 3. MAIN RESULT

We give the coefficients inequality for the class $K_s(A,B,s,t)$.

**Theorem 3.1:** *Let* $f\in K_s(A,B,s,t)$, *then for* $n\geq 1$,

$$|\,a_n\,| \leq \frac{\alpha}{n}\left(1+\frac{|\,u_n\,|}{|\,(n-u_n)\,|}\right)\left[1+\alpha\sum_{i_1=2}^{n-1}\frac{|\,u_{i_1}\,|}{|\,(i_1-u_{i_1})\,|} + \alpha^2\sum_{i_2>i_1}^{n-1}\ \sum_{i_1=2}^{n-2}\frac{|\,u_{i_1}u_{i_2}\,|}{|\,(i_1-u_{i_1})\,||\,(i_2-u_{i_2})\,|} + \ldots + \alpha^{n-2}\prod_{i=2}^{n-1}\frac{|\,u_i\,|}{|\,(i-u_i)\,|}\right], \tag{9}$$

where $\alpha = A-B$ and $u_i = \dfrac{(s^i - t^i)}{(s-t)}$.

**Proof:** Since $g\in S_s^*(A,B,s,t)$, this implies that $(s-t)zg'(z) = [g(sz)-g(tz)]K(z), z\in\Delta$, with $\mathrm{Re}\{K(z)\}>0$, where $K(z) = 1+c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ now equating coefficient of above equation, we get

$$b_2(2-u_2) = c_1 \tag{10}$$

$$b_3(3-u_3)=c_2+\frac{c_1^2u_2}{(2-u_2)} \tag{11}$$

$$b_4(4-u_4)=c_3+\frac{c_1c_2u_2}{(2-u_2)}+\frac{c_1c_2u_3}{(3-u_3)}+\frac{c_1^3u_2u_3}{(2-u_2)(3-u_3)} \tag{12}$$

$$b_5(5-u_5)=c_4+\frac{c_1c_3u_2}{(2-u_2)}+\frac{c_2^2u_3}{(3-u_3)}+\frac{c_1c_3u_4}{(4-u_4)}+\frac{c_1^2c_2u_2u_3}{(2-u_2)(3-u_3)}+\frac{c_1^2c_2u_2u_4}{(2-u_2)(4-u_4)}$$
$$+\frac{c_1^2c_2u_3u_4}{(3-u_3)(4-u_4)}+\frac{c_1^4u_2u_3u_4}{(2-u_2)(3-u_3)(4-u_4)}. \tag{13}$$

Now from (4), we get,

$$(s-t)zf'(z)=[g(sz)-g(tz)]P(z) \tag{14}$$

using the value from (5) and (6), we get

$$\begin{aligned} &z+2a_2z^2+3a_3z^3+\ldots+2na_{2n}z^{2n}+\left(2n+1\right)a_{2n+1}z^{2n+1}+\ldots \\ &=\left[z+b_2z^2u_2+b_3z^3u_3+\ldots+b_{2n}z^{2n}u_{2n}+b_{2n+1}z^{2n+1}u_{2n+1}+\ldots\right] \\ &\quad .(1+p_1z+p_2z^2+\ldots+p_{2n}z^{2n}+p_{2n+1}z^{2n+1}+\ldots). \end{aligned} \tag{15}$$

Equating the coefficients of various powers of z, we have,

$$2a_2=p_1+u_2b_2, \tag{16}$$

$$3a_3=b_3u_3+p_1b_2u_2+p_2, \tag{17}$$

$$4a_4=b_4u_4+p_1b_3u_3+p_2b_2u_2+P_3, \tag{18}$$

$$5a_5=b_5u_5+p_1b_4u_4+p_2b_3u_3+p_3b_2u_2+P_4, \tag{19}$$

on the similar manner, we can have

$$na_n=b_nu_n+b_{n-1}u_{n-1}p_1+\ldots+b_2u_2p_{n-2}+P_{n-1}, \tag{20}$$

where, $u_i=\dfrac{\left(s^i-t^i\right)}{\left(s-t\right)}$.

Easily using Lemma 2.1 and Lemma 2.2 in (16) and (17) respective, we get

$$2|a_2|\le\alpha\left[1+\frac{|u_2|}{|\left(2-u_2\right)|}\right], \tag{21}$$

$$3|a_3|\le\alpha\left[1+\frac{|u_3|}{|\left(3-u_3\right)|}\right]\left\{1+\alpha\frac{|u_2|}{|\left(2-u_2\right)|}\right\}, \tag{22}$$

Similarly using Lemma 2.1 and Lemma 2.2 in (18) and (19) respective, we get

$$4|a_4|\le\alpha\left[1+\frac{|u_4|}{|\left(4-u_4\right)|}\right]\left\{1+\alpha\left(\frac{|u_2|}{|\left(2-u_2\right)|}+\frac{|u_3|}{|\left(3-u_3\right)|}\right)+\alpha^2\left(\frac{|u_2u_3|}{|\left(2-u_2\right)||\left(3-u_3\right)|}\right)\right\},$$

(23)

$$5|a_5| \le \alpha\left[1+\frac{|u_5|}{|(5-u_5)|}\right]\left\{\begin{array}{l} 1+\alpha\left(\frac{|u_2|}{|(2-u_2)|}+\frac{|u_3|}{|(3-u_3)|}+\frac{|u_4|}{|(4-u_4)|}\right) \\ +\alpha^2\left(\frac{|u_2u_3|}{|(2-u_2)||(3-u_3)|}+\frac{|u_2u_4|}{|(2-u_2)||(4-u_4)|}+\frac{|u_3u_4|}{|(3-u_3)||(4-u_4)|}\right) \\ +\alpha^3\frac{|u_2u_3u_4|}{|(2-u_2)||(3-u_3)||(4-u_4)|} \end{array}\right\}. \tag{24}$$

It follows that from above equations Theorem 3.1 holds for n = 2, 3, 4 and 5. Now by Mathematical Induction, we can easily prove Theorem 3.1.

From (20) and Lemma 2.1 and Lemma 2.2, we get,

$$|a_n| \le \frac{\alpha}{n}\left(1+\frac{|u_n|}{|(n-u_n)|}\right)\left[1+\alpha\sum_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}+\alpha^2\sum_{i_2>i_1}^{n-1}\sum_{i_1=2}^{n-2}\frac{|u_{i_1}u_{i_2}|}{|(i_1-u_{i_1})||(i_2-u_{i_2})|}+\ldots+\alpha^{n-2}\prod_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}\right], \tag{25}$$

where $\alpha = A-B$ and $u_i = \frac{(s^i-t^i)}{(s-t)}$.

**Corollary 3.2:** Let $f \in K_s(A,\ B,\ t)$, then for n ≥ 1,

$$|a_n| \le \frac{\alpha}{n}\left(1+\frac{|u_n|}{|(n-u_n)|}\right)\left[1+\alpha\sum_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}+\alpha^2\sum_{i_2>i_1}^{n-1}\sum_{i_1=2}^{n-2}\frac{|u_{i_1}u_{i_2}|}{|(i_1-u_{i_1})||(i_2-u_{i_2})|}+\ldots+\alpha^{n-2}\prod_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}\right], \tag{26}$$

where $\alpha = A-B$ and $u_i = \frac{(1-t^i)}{(1-t)}$.

**Proof:** Let $S_s^*(A,B,t)$, the class of functions of the form (1) and satisfying the condition

$$\frac{(1-t)zf'(z)}{f(z)-f(zt)} \prec \phi(z);|t|\le 1,\ t\ne 1, \text{ where } \varphi(z)=(1+Az)/(1+Bz),(-1\le B< A\le 1)$$

then after solving we get the required result,

$$|a_n| \le \frac{\alpha}{n}\left(1+\frac{|u_n|}{|(n-u_n)|}\right)\left[1+\alpha\sum_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}+\alpha^2\sum_{i_2>i_1}^{n-1}\sum_{i_1=2}^{n-2}\frac{|u_{i_1}u_{i_2}|}{|(i_1-u_{i_1})||(i_2-u_{i_2})|}+\ldots+\alpha^{n-2}\prod_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}\right], \tag{27}$$

where $\alpha = A-B$ and $u_i = \frac{(1-t^i)}{(1-t)}$.

**SPECIAL CASES**

i. On putting s = 1 and t = –1 in Theorem 3.1, we get the known result [5].
ii. On the same manner if we put t = –1 in Corollary 3.2, we get the same result [5].

 

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